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# GENERALIZED CAPACITIES AND BOOLEAN FUZZY LOGIC

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**Abstract.** We propose an extension principle for fuzzy logic to obtain a Boolean logic able to manage incomplete and partially inconsistent information. Such a logic is strictly related with an extended notion of belief measure. The aim is to give suitable tools to design expert systems able to manage information probabilistic in nature.

**Keywords.** Boolean logic, Fuzzy logic, extension principle, approximate reasoning, capacities.

## Introduction.

Assume that $\mathcal{S}$ is a (crisp) deduction apparatus in Hilbert style defined by a set $A$ of logical axioms and a set $Ir$ of inference rules. Then, given an operation $\odot$ in the real interval [0,1] satisfying suitable properties, an extension principle enables us to extend $\mathcal{S}$ to a fuzzy deduction apparatus $\mathcal{S}^{\odot}$ whose fuzzy set of logical axioms is again $A$ and whose fuzzy inference rules are obtained by a suitable extension of the inference rules of $\mathcal{S}$ via the operation $\odot$ (see [1], [7], [8]).

In this paper we extend such an approach by considering any complete lattice $L$ equipped with an operation $\odot$ which is distributive with respect to the finite and infinite joins. In particular, we are interested to the extension of the classical propositional calculus in the case $L$ is a complete Boolean algebra $\boldsymbol{B}$ and $\odot$ is the usual meet operation. In such a case we obtain a *"Boolean"* logic rather different from the truth-functional Boolean logics usually considered in literature. Indeed, in this logic the theories (i.e. fuzzy subsets of formulas closed under deductions) are Boolean valuations which are not truth-functional, in general. As a matter of fact, these theories are very similar to the necessity measures and only the complete and totally consistent theories coincide with the truth-functional valuations in $\boldsymbol{B}$. Again, by considering a finitely additive probability in $\boldsymbol{B}$, we relate the theories of the proposed logic with the (generalized) belief measures as defined in [5] and [13]. In such a way we obtain an inferential apparatus, probabilistic in nature, in which both incompleteness and inconsistency are admitted. Finally, we consider the *"concrete"* examples based on a possible worlds semantics, i.e. we set $\boldsymbol{B} = P(W)$ where $W$ is a set whose elements we interpret as "possible worlds" or *"past cases"*. As sketched in [4], this enables us to design expert systems able to give information about an actual case by referring to stored information about similar past cases.

## 1. Preliminaries.

Let $L = (L, \wedge, \vee, 0, 1)$ be any complete and distributive lattice. We say that an element $x$ in $L$ is *complemented* if $x' \in L$ exists such that $x\vee x' = 1$ and $x\wedge x' = 0$. In such a case $x'$ is called the *complement of* $x$ and it is denoted by $-x$. If $x$ is a complemented element, then we write $x\rightarrow y$ to denote the element $-x\vee y$. We indicate by $B(L)$ the class of complemented elements of $L$. Such a class defines a Boolean algebra, namely the largest Boolean sub-algebra of $L$ (see [3]). Trivially, if $L$ is totally ordered, then $B(L) = \{0,1\}$.

**Proposition 1.1.** Given any $\lambda\in L$, $[\lambda,1]$ is a complete lattice whose zero element is $\lambda$. If $x$ is a complemented element of $L$, and $x\in[\lambda,1]$, then $x\rightarrow\lambda$ is the complement of $x$ in $[\lambda,1]$. Consequently, $B(L)\cap[\lambda,1] \subseteq B([\lambda,1])$ while $B([\lambda,1]) = B(L)\cap[\lambda,1]$ if and only if $\lambda$ is complemented. In particular, if $L$ is a Boolean algebra, $[\lambda,1]$ is a Boolean algebra, too.

*Proof.* Observe that if $x$ is complemented and $x\in[\lambda,1]$, then $x\wedge(-x\vee\lambda) = (x\wedge -x)\vee(x\wedge\lambda) = \lambda$ and $x\vee(-x\vee\lambda) = (x\vee -x)\vee\lambda = 1$. Also, assume that $\lambda$ is complemented and that $x\in B([\lambda,1])$. Then, if $x'$ is the complement of $x$ in $[\lambda,1]$, we claim that $x'\wedge\text{-}\lambda$ is the complement of $x$ in $L$ and therefore that $x \in B(L)\cap[\lambda,1]$. Indeed, since $-\lambda \geq -x$, it is $-\lambda\vee x \geq -x\vee x = 1$, and therefore

$$(x'\wedge\text{-}\lambda)\vee x = (x'\vee x)\wedge(-\lambda\vee x) = -\lambda\vee x = 1.$$

Moreover, $(x'\wedge\text{-}\lambda)\wedge x = (x'\wedge x)\wedge\text{-}\lambda = \lambda\wedge\text{-}\lambda = 0$.This proves that $B([\lambda,1]) = B(L)\cap[\lambda,1]$. Conversely, assume that $B([\lambda,1]) = B(L)\cap[\lambda,1]$. Then, since $\lambda\in B([\lambda,1])$, we have that $\lambda\in B(L)$ and therefore that $\lambda$ is complemented. □

Given a set $S$ and a complete lattice $L = (L, \wedge, \vee, 0, 1)$, an *L-subset* or *fuzzy subset* of $S$ is any map $s : S \to L$ from $S$ to $L$. For any $x \in S$, we interpret $s(x)$ as *the membership degree* of $x$ in $s$. Given two $L$-subsets $s$ and $s'$, we say that $s$ is *contained* in $s'$, in brief $s \subseteq s'$, if $s(x) \leq s'(x)$ for every $x$ in $S$. The *union* and the *intersection* of two $L$-subsets $s$ and $s'$ are defined by setting, for any $x\in S$, $(s\cup s')(x) = s(x)\vee s'(x)$ and $(s\cap s')(x) = s(x)\wedge s'(x)$, respectively. More generally, we define the union and the intersection of a family $(s_i)_{i\in I}$ of $L$-subsets by setting, for any $x \in S$, $(\cup_{i\in I} s_i)(x) = Sup_{i\in I} s_i(x)$ and $(\cap_{i\in I} s_i)(x) = Inf_{i\in I} s_i(x)$, respectively. Under these operations the class $L^S$ of all $L$-subsets of $S$ is a complete lattice. Indeed, such a lattice is the direct power of $L$ with index set $S$. Also, $L^S$ is an extension of the Boolean algebra $P(S)$. In fact, we can associate any classical subset $X$ of $S$ with its characteristic function, i.e., the $L$-subset $c_X$ defined by setting $c_X(x) = 1$ if $x \in X$, and $c_X(x) = 0$ otherwise. The resulting map is an injective homomorphism from the lattice $P(S)$ into the lattice $L^S$. This enables us to identify $P(S)$ with $\{0,1\}^S$. In other words, if we call *crisp* the $L$-subsets assuming only the Boolean values 0 and 1, we can identify the subsets of $S$ with the crisp $L$-subsets. In particular, we identify $\varnothing$ with the map constantly equal to 0 and $S$ with the map constantly equal to 1. Given an $L$-subset $s : S \to L$ of $S$ and $\lambda\in L$ we call *λ-cut* the classical subset

$$C(s,\lambda) = \{x\in S : s(x)\geq\lambda\}.$$

Since $s(x) = Sup\{\lambda\in L : x \in C(s,\lambda)\}$, any $L$-subset is characterized by its family of cuts.

## 2. Fuzzy inferential apparatus

Let $\mathcal{F}$ be a set whose elements we call *formulas*, then we define a (crisp) *Hilbert system* as a pair $\mathcal{S} = (A, Ir)$ where $A$ is a subset of $\mathcal{F}$, the *set of logical axioms*, and $Ir$ is a set of inference rules. In turn, an *inference rule* is a partial $n$-ary operation $r$ on $\mathcal{F}$ whose domain we denote by $Dom(r)$. We say that a set $T$ of formulas is *closed with respect to the inference rule r* if, for any $x_1,...,x_n \in \mathcal{F}$ such that $(x_1,...,x_n) \in Dom(r)$,

$$x_1 \in T, \ldots ,x_n \in T \Rightarrow r(x_1,...,x_n) \in T.$$

We say that $T$ is a *theory* if $T$ contains the set $A$ of logical axioms and it is closed with respect to all the inference rules. A *proof π of a formula α under a set X of hypothesis* is a sequence $\alpha_1,...,\alpha_m$ of formulas such that $\alpha_m = \alpha$ and, for any $1\leq i \leq m$, either

(i) $\alpha_i$ is an element in $A$, or
(ii) $\alpha_i$ is an element in $X$, or
(iii) $\alpha_i = r(\alpha_{s(1)},\ldots, \alpha_{s(n)})$ where $r$ is an inference rule and $s(1)<i, ..., s(n)<i$.

If a proof $\pi$ of $\alpha$ under hypothesis $X$ exists, then we write $X \vdash \alpha$. We say that $\alpha$ is *logically equivalent* with $\alpha'$ provided that $\alpha \vdash \alpha'$ and $\alpha' \vdash \alpha$ and in such a case we write $\alpha \equiv \alpha'$. Also, we call *deduction operator* associated with $\mathcal{S}$ the operator $\mathcal{D} : \mathcal{P}(\mathcal{F}) \to \mathcal{P}(\mathcal{F})$ defined by setting,

$$\mathcal{D}(X) = \{\alpha \in \mathcal{F} : X \vdash \alpha\}.$$

In other words, $\mathcal{D}(X)$ is the set of formulas we can prove from $X$. It is easy to prove that $\mathcal{D}$ is a compact closure operator and that the fixed points of $\mathcal{D}$ coincide with the theories. In particular, $\mathcal{D}(X)$ is a theory we call *the theory generated by X*.

We extend these definitions in accordance with J. Pavelka [10] by defining an *L-Hilbert system* as a pair $\mathcal{S} = (a, Ir)$ where $a$ is an $L$-subset of $\mathcal{F}$, the *L-subset of logical axioms*, and $Ir$ is a set of $L$-inference rules. In turn, an *L-inference rule* is a pair $r = (r',r'')$, where $r'$ is a partial $n$-ary operation on $\mathcal{F}$ whose domain we denote by $Dom(r)$ and $r''$ is an $n$-ary operation on $L$ preserving the least upper bound in each variable, i.e. satisfying the condition $r''(x_1,..., Sup_{i\in I}\, y_i, ..., x_n) = Sup_{i\in I}\, r''(x_1, ..., y_i, ..., x_n)$. We indicate an application of an inference rule $r$ by the picture

$$\frac{\alpha_1,...,\alpha_n}{r'(\alpha_1,...,\alpha_n)} \quad ; \quad \frac{\lambda_1,...,\lambda_n}{r''(\lambda_1,...,\lambda_n)}$$

whose meaning is that:
IF you know that $\alpha_1,\ldots,\alpha_n$ are true (at least) to the degree $\lambda_1,\ldots,\lambda_n$
THEN $r'(\alpha_1,\ldots,\alpha_n)$ is true (at least) at level $r''(\lambda_1,\ldots,\lambda_n)$.
We say that an $L$-subset $s$ of formulas is *closed with respect to an L-rule* $r = (r',r'')$ if

$$s(r'(\alpha_1,\ldots,\alpha_n)) \geq r''(s(\alpha_1),\ldots,s(\alpha_n)).$$

We call *theory* any fuzzy subset $\tau$ of formulas containing the $L$-subset of logical axioms and closed with respect to the $L$-inference rules. This extends the classical notion of a theory as a set of formulas closed under deductions. A *proof* $\pi$ *of a formula* $\alpha$ is a sequence $\alpha_1,...,\alpha_m$ of formulas such that $\alpha_m = \alpha$, together with the related *"justifications"*. This means that, for any formula $\alpha_i$, we must specify whether

(i) $\alpha_i$ is assumed as a logical axiom; or
(ii) $\alpha_i$ is assumed as a hypothesis; or
(iii) $\alpha_i$ is obtained by a rule (in this case we must indicate also the rule and the formulas from $\alpha_1,...,\alpha_{i-1}$ used to obtain $\alpha_i$).

Observe that we have only two proofs of $\alpha$ whose length is equal to 1: the formula $\alpha$ with the justification that $\alpha$ is assumed as a logical axiom and the formula $\alpha$ with the justification that $\alpha$ is assumed as a hypothesis. Moreover, as in the classical case, for any $i \leq m$, the initial segment $\alpha_1,...,\alpha_i$ is a proof of $\alpha_i$ we denote by $\pi(i)$. Let $v : \mathbb{F} \to L$ be a fuzzy subset of formulas we call *initial valuation* and let $\pi$ be a proof of $\alpha$. Then we define the *valuation* $Val(\pi,v)$ *of* $\pi$ *with respect to* $v$ by induction on the length $m$ of $\pi$ by setting

$$Val(\pi,v) = \begin{cases} a(\alpha_m) & \text{if } \alpha_m \text{ is assumed as a logical axiom,} \\ v(\alpha_m) & \text{if } \alpha_m \text{ is assumed as a hypothesis,} \\ r''(Val(\pi(i(1)),v),\ldots,Val(\pi(i(n)),v)) & \text{if } \alpha_m = r'(\alpha_{i(1)},\ldots,\alpha_{i(n)}) \end{cases}$$

where, $1\leq i(1)<m,\ldots,1\leq i(n)<m$. The meaning we assign to $Val(\pi,v)$ is that: *given the information* $v$, $\pi$ *assures that* $\alpha$ *holds at least at level* $Val(\pi,v)$. Differently from the crisp case, the justifications are necessary since different justifications of the same formula give rise to different valuations. Also, different proofs of the same formula can lead to different valuations and this suggests the following definition.

**Definition 2.3**. Given an $L$-Hilbert system $\mathcal{S}$, we call *deduction operator* associated with $\mathcal{S}$ the operator $\mathcal{D} : L^{\mathbb{F}} \to L^{\mathbb{F}}$ defined by setting,

$$\mathcal{D}(v)(\alpha) = Sup\{Val(\pi,v) : \pi \text{ is a proof of } \alpha\}, \tag{2.1}$$

for every initial valuation $v$ and every formula $\alpha$.

The meaning of $\mathcal{D}(v)(\alpha)$ is still

*given the information v, we may prove that* $\alpha$ *holds at least at level* $\mathcal{D}(v)(\alpha)$,

but we have also that

$\mathcal{D}(v)(\alpha)$ *is the best possible valuation we can draw from the information v.*

The following theorem shows some basic properties of the operator $\mathcal{D}$ (see [10] and [8]).

**Theorem 2.4.** *The deduction operator* $\mathcal{D} : L^{\mathbb{F}} \to L^{\mathbb{F}}$ *of an L-Hilbert system is a continuous closure operator, i.e., for every* $v, v' \in L^{\mathbb{F}}$, *and* $(v_n)_{n\in N}$ *order-preserving sequence of L-subsets*

i) $\mathcal{D}(v) \supseteq v$ ;
ii) $v \subseteq v' \Rightarrow \mathcal{D}(v) \subseteq \mathcal{D}(v')$ ;
iii) $\mathcal{D}(\mathcal{D}(v)) = \mathcal{D}(v)$ ;
iv) $\mathcal{D}(\cup_{n\in N}v_n) = \cup_{n\in N}\mathcal{D}(v_n)$.

Moreover, the fixed points of $\mathcal{D}$ coincide with the theories.

Differently from the crisp Hilbert systems, fuzzy Hilbert systems are able to manage inconsistency.

**Definition 2.5.** We call *degree of inconsistency* of an initial valuation $v$ the value

$$Inc(v) = Inf\{\mathcal{D}(v)(\alpha) : \alpha \in \mathbb{F}\}. \tag{2.2}$$

We say that $v$ is totally consistent (totally inconsistent) if $Inc(v) = 0$ (if $Inc(v) = 1$, respectively).

In accordance, we call *totally inconsistent theory* the map in $\mathbb{F}$ constantly equal to 1. Then, a valuation $v$ is totally inconsistent if $D(v)$ collapses into the totally inconsistent theory.

In [10], Pavelka defines an *L-semantics* or *fuzzy semantics* any class $\mathcal{M}$ of $L$-subsets of $\mathbb{F}$. An element $m \in \mathcal{M}$ is a *model* of an initial valuation $v$ provided that $m \supseteq v$. In such a case we write $m \models v$. Any $L$-semantics is associated with a *logical consequence operator* $Lc : L^{\mathbb{F}} \to L^{\mathbb{F}}$ defined by setting, for any $L$-subset of formulas $v$,

$$Lc(v) = \cap\{m \in \mathcal{M} : m \models v\}. \tag{2.3}$$

**Definition 2.6.** An *L-logic* or *fuzzy logic* is an $L$-semantics $\mathcal{M}$ together with an $L$-Hilbert system such that $Lc = \mathcal{D}$, i.e. the logical consequence operator coincides with the deduction operator.

## 3. Canonical extensions of the classical propositional calculus

An interesting class of *L-Hilbert systems* is obtained as follows. We assume that the lattice $L$ is equipped with an associative, commutative operation $\odot$ satisfying the identity $1\odot x = x$ and the distributivity condition

$$(Sup_{i\in I}x_i)\odot x = Sup_{i\in I}(x_i\odot x)$$

for any family $(x_i)_{i\in I}$ of elements in $L$ and $x\in L$. Trivially, $\odot$ is order-preserving and $x\odot 0 = 0$. Typical examples are obtained by assuming that $L$ is the interval [0,1] and $\odot$ is a continuous triangular norm. For instance, we can consider the usual product or the Lukasiewicz norm.

**Definition 3.1.** Let $(A,Ir)$ be a crisp Hilbert-system. Then given an $n$-ary crisp rule $r\in Ir$, we call *canonical* $\odot$*-extension of* $r$ the $L$-rule $(r',r'')$ defined by setting $r' = r$ and $r''(\lambda_1,\dots,\lambda_n) = \lambda_1\odot\dots\odot\lambda_n$. We call $\odot$*-canonical extension* of $(A,Ir)$ the fuzzy Hilbert system $(c_A,Ir^{\odot})$ such that :

- the $L$-subset of logical axioms is the characteristic function $c_A$ of $A$
- the set $Ir^{\odot}$ of inference rules is the set of canonical $\odot$-extensions of the rules in $Ir$.

If we denote by $\mathcal{D}$ the deduction operator of $(A,Ir)$, then we denote by $\mathcal{D}^{\odot}$ the deduction operator of the $\odot$*-canonical extension* of $(A,Ir)$. In this paper we are interested to the extensions of the classical propositional calculus, i.e. we assume that

- $\mathbb{F}$ is the set of formulas of the propositional calculus whose logical connectives are $\wedge$, $\vee$, and $\neg$ (we denote by $\wedge$ and $\vee$ both the logical connectives and the lattice operations in $L$) ;

- $c_A$ is characteristic function of the set of all the tautologies ;

- the (only) inference rule is the $L$-extension of the *Modus Ponens* rule

$$\frac{\alpha\,,\,\alpha\to\beta}{\beta} \quad ; \quad \frac{x\,,\,y}{x\odot y}$$

We have the following theorem extending a theorem proved in [7].

**Theorem 3.1.** *Let* $\mathcal{D}^{\odot} : L^{\mathbb{F}}\to L^{\mathbb{F}}$ *be the deduction operator of the* $\odot$*-canonical extension of the classical propositional calculus. Then, for any* $v\in L^{\mathbb{F}}$ *and* $\alpha\in\mathbb{F}$,

$$\mathcal{D}^{\odot}(v)(\alpha) = \begin{cases} 1 & \textit{if } \alpha \textit{ is a tautology,} \\ Sup\{v(\alpha_1)\odot...\odot v(\alpha_n) : \alpha_1, \dots, \alpha_n \vdash \alpha\} & \textit{otherwise.} \end{cases} \tag{3.1}$$

*Proof.* If $\alpha$ is a tautology is evident that $\mathcal{D}^{\odot}(v)(\alpha) = 1$. Otherwise, set $h = Sup\{v(\alpha_1)\odot\ldots\odot v(\alpha_1) : \alpha_1,\ldots,\alpha_n \vdash \alpha\}$ and assume that $\alpha_1,...,\alpha_n$ are formulas such that $\alpha_1,...,\alpha_n \vdash \alpha$.. Then, by the Deduction Theorem, $\alpha_1 \to (... (\alpha_n \to \alpha))$ is a tautology, and therefore the sequence

| | |
|---|---|
| $\alpha_1$, | $v(\alpha_1)$, |
| ... , | ... |
| $\alpha_n$, , | $v(\alpha_n)$, |
| $\alpha_1 \to (... (\alpha_n \to \alpha))$ | 1, |
| $\alpha_2 \to (... (\alpha_n \to \alpha))$, | $1\odot v(\alpha_1) = v(\alpha_1)$, |
| . . . | . . . |
| $\alpha_n \to \alpha$ | $v(\alpha_1)\odot...\odot v(\alpha_{n-1})$, |
| $\alpha$ | $v(\alpha_1)\odot...\odot v(\alpha_n)$. |

is a proof $\pi$ of $\alpha$ such that $Val(\pi,v) = v(\alpha_1)\odot...\odot v(\alpha_n)$. This proves that $v(\alpha_1)\odot...\odot v(\alpha_n) \le D^{\odot}(v)(\alpha)$ and therefore $h \le D^{\odot}(v)(\alpha)$. To prove that $h \ge D^{\odot}(v)(\alpha)$, it is sufficient to prove that $h \ge Val(\pi, v)$ for any proof $\pi$ of $\alpha$ in the canonical extension. Indeed, assume that $\alpha_1,...,\alpha_h$ are the hypotheses used in $\pi$. Then, $n(1), ..,n(h)$ exist such that $Val(\pi,v) = v(\alpha_1)^{n(1)} \odot...\odot v(\alpha_h)^{n(h)}$. By observing that $\alpha_1,...,\alpha_h \vdash \alpha$ and that

$$v(\alpha_1)^{n(1)}\odot...\odot v(\alpha_h)^{n(h)} \le v(\alpha_1)\odot...\odot v(\alpha_h),$$

we conclude that $Val(\pi,v) \le h$. □

We call $\odot$-*theory* any theory of the $\odot$-canonical extension of the classical propositional calculus, i.e. any fixed point of $\mathcal{D}^{\odot}$. Obviously, an $L$-subset $\tau$ of formulas is a $\odot$-theory iff

j) $\tau(\alpha) = 1$ for any tautology $\alpha$

jj) $\tau(\beta) \ge \tau(\alpha)\odot\tau(\alpha\to\beta)$ for any $\alpha$ and $\beta$ in $\mathbb{F}$.

It is evident that the crisp $\odot$-theories coincide with the (characteristic function of the) classical theories. The following theorem summarizes the main properties of the $\odot$-theories. We say that an $L$-subset of formulas $v$ is *compatible* with the logical equivalence $\equiv$ if

$$\alpha \equiv \beta \Rightarrow v(\alpha) = v(\beta).$$

**Proposition 3.2.** *Let $\tau$ be a $\odot$-theory. Then the following properties hold true:*

i) $\alpha \vdash \beta \Rightarrow \tau(\alpha) \le \tau(\beta)$;

ii) *$\tau$ is compatible with the logical equivalence*;

iii) $\tau(\alpha)\wedge\tau(\beta) \ge \tau(\alpha\wedge\beta) \ge \tau(\alpha)\odot\tau(\beta)$;

iv) $\tau(\alpha\vee\beta) \ge \tau(\alpha)\vee\tau(\beta)$.

*Proof.* Propositions i), ii) and iv) are evident. To prove iii), observe that, since $\alpha\wedge\beta \vdash \alpha$ and $\alpha\wedge\beta \vdash \beta$, we have that $\tau(\alpha\wedge\beta) \le \tau(\alpha)$ and $\tau(\alpha\wedge\beta)\le\tau(\beta)$ and therefore that $\tau(\alpha\wedge\beta)\le\tau(\alpha)\wedge\tau(\beta)$. Moreover, since $\beta \vdash \alpha\to(\alpha\wedge\beta)$, we have that $\tau(\beta)\le\tau(\alpha\to(\alpha\wedge\beta))$, and therefore

$$\tau(\alpha\wedge\beta) \ge \tau(\alpha)\odot\tau(\alpha\to(\alpha\wedge\beta))\ge\tau(\alpha)\odot\tau(\beta).$$ □

The following proposition characterizes the $\odot$-theories.

**Theorem 3.3.** *An L-subset $\tau : \mathbb{F}\to L$ is a $\odot$-theory iff, for any $\alpha$ and $\beta$,*

j) $\alpha \vdash \beta \Rightarrow \tau(\alpha) \le \tau(\beta)$ ;

jj) *$\alpha$ tautology* $\Rightarrow \tau(\alpha) = 1$ ;

jjj) $\tau(\alpha\wedge\beta) \ge \tau(\alpha)\odot\tau(\beta)$.

*Proof.* By Proposition 3.2 any ⊙-theory satisfies j), jj) and jjj). Conversely, assume that $\tau$ satisfies j) and jjj). Then, since

$$\tau(\alpha)\odot\tau(\alpha\to\beta) \le \tau(\alpha\wedge\alpha\to\beta) = \tau(\alpha\wedge\beta) \le \tau(\beta),$$

$\tau$ is closed with respect to *Modus Ponens* rule. Then, by jj), $\tau$ is a ⊙-theory. □

The following proposition characterizes the degree of inconsistency of an initial valuation.

**Proposition 3.4.** *Let v be an initial valuation. Then,*

$$Inc(v) = Sup\{v(\alpha_1)\odot...\odot v(\alpha_n) : \alpha_1, \dots, \alpha_n \text{ are inconsistent}\} \quad (3.2)$$

*and, given any contradiction* $\phi$,

$$Inc(v) = \mathcal{D}^{\odot}(v)(\phi), \quad (3.3)$$

*Proof.* (3.2) is evident. To prove (3.3) observe that, since $\phi \vdash \alpha$ for any formula $\alpha$, from ii) of Proposition 3.2, we have that $\mathcal{D}^{\odot}(\phi) \le \mathcal{D}^{\odot}(\alpha)$. □

The following proposition shows that there are theories with any inconsistency degree.

**Proposition 3.5.** *Let* $f : L \to L$ *be an order preserving mapping such that* $f(1) = 1$ *and* $f(x\odot y) \ge f(x)\odot f(y)$. *Then, for any ⊙-theory* $\tau$, *the L-subset* $f(\tau) = f\circ\tau$ *is a ⊙-theory such that*

$$Inc(f(\tau)) = f(Inc(\tau)).$$

*Consequently, for any* $\lambda \in L$, *a ⊙-theory exists whose inconsistency degree is* $\lambda$.

*Proof.* Assume that $\alpha$ is a tautology. Then $f(\tau)(\alpha) = f(\tau(\alpha)) = f(1) = 1$. Also, since $\tau(\beta) \ge \tau(\alpha)\odot\tau(\alpha\to\beta)$, we have also that $f(\tau(\beta)) \ge f(\tau(\alpha)\odot\tau(\alpha\to\beta)) \ge f(\tau(\alpha))\odot f(\tau(\alpha\to\beta))$. This proves that $f(\tau)$ is a theory. Obviously, given any contradiction $\phi$, we have that $Inc(f(\tau)) = f(\tau(\phi)) = f(Inc(\tau))$.

To prove that a ⊙-theory exists whose inconsistency degree is $\lambda$, let $f : L \to L$ be the mapping defined by setting $f(x) = 1$ if $x = 1$ and $f(x) = \lambda$ otherwise. Then it is easy to prove that $f$ is order-preserving and that $f(x\odot y) \ge f(x)\odot f(y)$. Let $\tau$ be the characteristic function of a classical consistent theory. Then $f(\tau)$ is an ⊙-theory such that $Inc(f(\tau)) = f(Inc(\tau)) = f(0) = \lambda$. □

**4. ∧-canonical extension**

In this section we assume that ⊙ is the meet operator ∧ in $L$ and therefore that the distributive law

$$(Sup_{i\in I}x_i)\wedge x = Sup_{i\in I}(x_i\wedge x)$$

holds. In such a case given a theory $\tau$, by iii) of Proposition 3.2, $\tau(\alpha)\wedge\tau(\beta) = \tau(\alpha\wedge\beta)$, while the equation $\tau(\alpha\wedge\beta) = \tau(\alpha)\odot\tau(\beta)$ fails, in general. In fact, assume that ⊙ is not idempotent, for example that $\lambda\odot\lambda \ne \lambda$. Also, consider a consistent classical theory $T$ and a formula $\alpha \notin T$. Then by setting $f(x) = x\vee\lambda$, by Proposition 3.5, $\tau = f\circ c_T$ is a theory such that $\tau(\alpha) = \lambda$. On the other hand, $\tau(\alpha\wedge\alpha) = \tau(\alpha) = \lambda \ne \lambda\odot\lambda = \tau(\alpha)\odot\tau(\alpha)$.

The following theorem shows that the ∧-theories are very similar to the necessity measures as defined in [2] or in [6].

**Theorem 4.1.** *An L-subset* $\tau : \mathbb{F} \to L$ *is a ∧-theory iff, for any* $\alpha$ *and* $\beta$,

i) *$\tau$ is compatible with the logical equivalence* ;
ii) *$\alpha$ tautology* $\Rightarrow \tau(\alpha) = 1$ ;
iii) $\tau(\alpha\wedge\beta) = \tau(\alpha)\wedge\tau(\beta)$.

*Proof.* By Proposition 3.2 any ∧-theory satisfies i), ii) and iii). Conversely, assume that $\tau$ satisfies i) and iii). Then, since $\tau(\alpha)\wedge\tau(\alpha\to\beta) = \tau(\alpha\wedge\alpha\to\beta) = \tau(\alpha\wedge\beta) = \tau(\alpha)\wedge\tau(\beta) \le \tau(\beta)$, $\tau$ is closed with respect to Modus Ponens rule. Thus, by ii), $\tau$ is a ∧-theory. □

It is also possible to characterize the ∧-theories by the related cuts.

**Theorem 4.2.** *An L-subset* $\tau : \mathbb{F} \to L$ *is a* $\wedge$*-theory iff, for any* $\lambda \in L$*, the* $\lambda$*-cut* $C(\tau,\lambda)$ *is a classical theory. Moreover,*

$$Inc(v) = Sup\{\lambda \in L : C(v,\lambda) \text{ is inconsistent}\}. \qquad (4.1)$$

*Proof.* Let $\tau$ be a $\wedge$-theory, $\lambda \in L$, and $f$ be the mapping such that $f(x) = 1$ if $x \geq \lambda$ and $f(x) = 0$ otherwise. Then $f$ is order preserving and such that $f(x \wedge y) \geq f(x) \wedge f(y)$ and therefore $f(\tau)$ is a $\wedge$-theory. Since $f(\tau)$ is the characteristic function of the cut $C(\tau,\lambda)$, such a cut is a theory.

Conversely, assume that each $C(\tau,\lambda)$ is a theory. Then, since $C(\tau,1)$ is a theory, for any tautology $\alpha$ we have that $\alpha \in C(\tau,1)$ and therefore that $\tau(\alpha) = 1$. Let $\alpha$ and $\beta$ be two formulas and set $\lambda = \tau(\alpha) \wedge \tau(\alpha \to \beta)$. Then, since $\alpha \in C(\tau,\lambda)$ and $\alpha \to \beta \in C(\tau,\lambda)$ we have that $\beta \in C(\tau,\lambda)$ and therefore that $\tau(\beta) \geq \lambda = \tau(\alpha) \wedge \tau(\alpha \to \beta)$. This proves that $\tau$ is a $\wedge$-theory.

To prove (4.1), observe that, by (3.3),

$$Inc(v) = Sup\{v(\alpha_1) \wedge \ldots \wedge v(\alpha_n) : \alpha_1, \ldots, \alpha_n \text{ is inconsistent}\}.$$

Now, if $\alpha_1,\ldots,\alpha_n$ are inconsistent formulas and $\lambda = v(\alpha_1) \wedge \ldots \wedge v(\alpha_n)$, then $C(v,\lambda)$ is inconsistent. This proves that $Inc(v) \leq Sup\{\lambda \in L : C(v,\lambda)$ is inconsistent$\}$. Conversely, assume that $C(v,\lambda)$ is inconsistent. Then an inconsistent set of formulas $\alpha_1, \ldots, \alpha_n$ exists such that $v(\alpha_1) \wedge \ldots \wedge v(\alpha_n) \geq \lambda$. This proves that $Inc(v) \geq Sup\{\lambda \in L : C(v,\lambda)$ is inconsistent$\}$. □

We extend the classical notion of completeness as follows.

**Definition 4.3**. Let $v : \mathbb{F} \to L$ be an initial valuation. We say that a formula $\alpha$ is *decidable in v* if,

$$\mathcal{D}^{\wedge}(v)(\alpha) \vee \mathcal{D}^{\wedge}(v)(\neg\alpha) = 1. \qquad (4.2)$$

and that $v$ is *complete* if any formula is decidable in $v$. We denote by $\mathcal{M}$ the class of all complete and totally consistent theories, by $\mathcal{M}^*$ the class of all complete theories.

Trivially, if $v$ is complete and $v' \sqsupseteq v$, then $v'$ is complete, too. Differently from classical logic, the notion of a complete theory is different form the notion of maximal theory. For example, let $L = [0,1]$ and let $\tau$ be the characteristic function of a complete theory (in classical propositional calculus). Then for any $\lambda \neq 1$ the fuzzy set $\tau \vee \lambda$ is a complete theory which is not maximal. Indeed, as a matter of fact, in such a logic no maximal theory exists.

The notion of completeness is related with the notion of truth-functional valuation.

**Definition 4.4.** A (*Boolean*) *truth-functional valuation in L* is a truth-functional valuation in the Boolean algebra $B(L)$, i.e. a map $m : \mathbb{F} \to B(L)$ such that

- $A_1$) $m(\alpha \wedge \beta) = m(\alpha) \wedge m(\beta)$ ;
- $A_2$) $m(\alpha \vee \beta) = m(\alpha) \vee m(\beta)$ ;
- $A_3$) $m(\neg\alpha) = -m(\alpha)$.

Observe that the logic associated with the Boolean truth-functional valuations is, in a sense, equivalent to the classical logic. More precisely, we have that

- $B_1$) $\alpha \equiv \beta \Leftrightarrow m(\alpha) = m(\beta)$ for any truth-functional $m$;
- $B_2$) $\alpha$ is a tautology $\Leftrightarrow m(\alpha) = 1$ for any truth-functional $m$;
- $B_3$) $\phi$ is a contradiction $\Leftrightarrow m(\phi) = 0$ for any truth-functional $m$.

These properties are immediate consequences of the fact that any Boolean algebra $\boldsymbol{B}$ is a sub-algebra of a suitable direct power of $\{0,1\}$. Then, given two terms $t_1$ and $t_2$, the equation $t_1 = t_2$ is satisfied by $\boldsymbol{B}$ iff it is satisfied by $\{0,1\}$.

**Theorem 4.5.** *The class* $\mathcal{M}$ *of complete totally consistent* $\wedge$*-theories coincides with the class of the truth-functional valuations.*

*Proof.* Let $\tau$ be a complete totally consistent $\wedge$-theory. Then, since for any formula $\alpha$, $\tau(\alpha)\vee\tau(\neg\alpha) = 1$, and $\tau(\alpha)\wedge\tau(\neg\alpha) = 0$, we have that $\tau(\alpha) \in B(L)$ and $\tau(\neg\alpha) = -\tau(\alpha)$ which proves $A_3$). To prove $A_2$, observe that,

$$\tau(\alpha\vee\beta) = \tau(\neg(\neg\alpha\wedge\neg\beta)) = -\tau(\neg\alpha\wedge\neg\beta)) = -(\tau(\neg\alpha)\wedge\tau(\neg\beta))) = -(-\tau(\alpha)\wedge-\tau(\beta)) = \tau(\alpha)\vee\tau(\beta).$$

Since $\tau$ is a $\wedge$-theory, $A_1$) holds and we can conclude that $\tau$ is truth-functional.

Conversely, let $m : \mathbb{F}\to B(L)$ be a truth-functional valuation. Then, i), ii) and iii) in Theorem 4.1 are satisfied, and therefore $m$ is a $\wedge$-theory. $B_3$) says that $m$ is totally consistent. Finally, since $m(\alpha)\vee m(\neg\alpha) = m(\alpha\vee\neg\alpha) = 1$, $m$ is complete. □

**Theorem 4.6.** *Let $\tau$ be a $\wedge$-theory. Then the following are equivalent*:

i) *$\tau$ is complete*;

ii) *$\tau$ is a truth-functional valuation in the lattice* $[Inc(\tau),1]$;

iii) $\tau(\alpha\vee\beta) = \tau(\alpha)\vee\tau(\beta)$ *for any $\alpha$ and $\beta$ in $\mathbb{F}$.*

*Proof.* i) $\Rightarrow$ ii) Observe that $\tau$ is a complete and totally consistent $\wedge$-theory in the lattice $[Inc(\tau),1]$ and therefore, by Theorem 4.5, $\tau$ is truth-functional in such a lattice.

ii)$\Rightarrow$ iii) Trivial.

iii) $\Rightarrow$ i) From iii), by setting $\beta = \neg\alpha$, we obtain that

$$\tau(\alpha)\vee\tau(\neg\alpha) = \tau(\alpha\vee\neg\alpha) = 1.$$ □

We can restate all the results in this section in terms of homomorphisms. Indeed, consider *the Boolean algebra of formulas* $(\mathbb{F}^*, \wedge, \vee, -, \mathbf{0},\mathbf{1})$, defined by setting

$$\mathbb{F}^* = \{[\alpha] : \alpha \in \mathbb{F}\}\ ,\ [\alpha] = \{\alpha' : \alpha' \equiv \alpha\},\ \ \mathbf{0} = \{\phi : \phi \text{ is a contradiction}\},\ \mathbf{1} = \{\alpha : \alpha \text{ is a tautology}\},$$

$$[\alpha]\wedge[\beta] = [\alpha\wedge\beta]\ \ ;\ \ [\alpha]\vee[\beta] = [\alpha\vee\beta]\ \ ;\ \ -[\alpha] = [\neg\alpha].$$

Notice that in the case an infinite number of propositional variables is admitted, such a Boolean algebra is not complete.

The class of $L$-subsets of formulas compatible with the logical equivalence is in a one-one correspondence with the class of maps from $\mathbb{F}^*$ to $L$, i.e. the class of $L$-subsets of $\mathbb{F}^*$. Indeed, if $v : \mathbb{F}\to L$ is a compatible $L$-subset, then we can define $v' : \mathbb{F}^*\to L$ by setting $v'([\alpha]) = v(\alpha)$ for any $\alpha\in \mathbb{F}$. Conversely, given a map $v' : \mathbb{F}^*\to L$, we can define a compatible map $v : \mathbb{F}\to L$ by setting $v(\alpha) = v'([\alpha])$ for any $\alpha\in\mathbb{F}$.

**Theorem 4.7.** *Let* $(\mathbb{F}^*,\wedge,\vee,-, \mathbf{0},\mathbf{1})$ *be the Boolean algebra of formulas. Then:*

i) *the $\wedge$-theories are in a one-one correspondence with the homomorphisms from* $(\mathbb{F}^*, \wedge, \mathbf{1})$ *into* $(L, \wedge, 1)$ ;

ii) *the totally consistent $\wedge$-theories are in a one-one correspondence with the homomorphisms from* $(\mathbb{F}^*, \wedge, \mathbf{0}, \mathbf{1})$ *into* $(L, \wedge, 0, 1)$ ;

iii) *the complete $\wedge$-theories are in a one-one correspondence with the homomorphisms from* $(\mathbb{F}^*, \wedge, \vee, \mathbf{1})$ *into* $(L, \wedge, \vee, 1)$ ;

iv) *the totally consistent and complete $\wedge$-theories are in a one-one correspondence with the homomorphisms from* $(\mathbb{F}^*, \wedge, \vee, -, \mathbf{0}, \mathbf{1})$ *to* $(B(L), \wedge, \vee, -, 0, 1)$.

*Proof.* See Theorems 4.1, 4.5 and 4.6. □

**5. Boolean extensions and completeness theorem**

In this section we will consider the case in which the lattice $L$ is a complete Boolean algebra we denote by $\boldsymbol{B}$. Note that in such a case the distributive law $x\wedge(Sup_{i\in I}x_i) = (Sup_{i\in I}x\wedge x_i)$ holds. At first, we are interested in some characterizations of the complete $\boldsymbol{B}$-theories.

**Proposition 5.1.** *Let $\lambda$ be any element in $\boldsymbol{B}$. Then the class of complete theories whose inconsistency degree is $\lambda$ coincides with the class of all the valuations $\tau$ such that $\tau(p_i) \geq \lambda$ for any propositional variable $p_i$ and*

*a)* $\tau(\alpha\wedge\beta) = \tau(\alpha)\wedge\tau(\beta)$ ;

*b)* $\tau(\alpha\vee\beta) = \tau(\alpha)\vee\tau(\beta)$ ;

*c)* $\tau(\neg\alpha) = \tau(\alpha)\rightarrow\lambda$.

*Proof.* The class of complete theories whose degree of inconsistency is $\lambda$ coincides with the class of complete completely consistent theories in $[\lambda,1]$. Thus, the proposition follows from Proposition 1.1 and Theorem 4.5. □

**Proposition 5.2.** *Let $\mathcal{M}_\lambda$ be the class of complete theories whose degree of inconsistency is $\lambda$. Then*

$$\mathcal{M}_\lambda = \{m\vee\lambda : m\in\mathcal{M}\}. \tag{5.1}$$

*Consequently, if $\mathcal{M}^*$ is the class of complete theories,*

$$\mathcal{M}^* = \{m\vee\lambda : m\in\mathcal{M},\ \lambda\in\boldsymbol{B}\}. \tag{5.2}$$

*Proof.* Let $\tau$ be a complete theory whose inconsistency degree is $\lambda$ and let $m\in\mathcal{M}$ be the truth-functional valuation such that $m(p_i) = \tau(p_i)$ for every propositional variable $p_i$. We will prove that, for every formula $\alpha$,

$$\tau(\alpha) = m(\alpha)\vee\lambda \tag{5.3}$$

and therefore that $\tau = m\vee\lambda$. We proceed by induction on the complexity of $\alpha$. Indeed, if $\alpha$ is the propositional variable $p_i$, then, since $m(p_i) = \tau(p_i) \geq \lambda$, (5.3) holds true. Assume that $\alpha$ and $\beta$ satisfy (5.3). Then, since

$$\tau(\alpha\wedge\beta) = \tau(\alpha)\wedge\tau(\beta) = (m(\alpha)\vee\lambda)\wedge(m(\beta)\vee\lambda) = (m(\alpha)\wedge m(\beta))\vee\lambda = m(\alpha\wedge\beta))\vee\lambda,$$

and

$$\tau(\alpha\vee\beta) = \tau(\alpha)\vee\tau(\beta) = (m(\alpha)\vee\lambda)\vee(m(\beta)\vee\lambda)$$
$$= (m(\alpha)\vee m(\beta))\vee\lambda = m(\alpha\vee\beta))\vee\lambda,$$

both $\alpha\wedge\beta$ and $\alpha\vee\beta$ satisfy (5.3). Since by c) of Proposition 5.1,

$$\tau(\neg\alpha) = (-\tau(\alpha))\vee\lambda = (-(m(\alpha)\vee\lambda))\vee\lambda$$
$$= (-m(\alpha)\wedge-\lambda)\vee\lambda = (m(\neg\alpha)\wedge-\lambda)\vee\lambda$$
$$= (m(\neg\alpha)\vee\lambda)\wedge(-\lambda\vee\lambda) = m(\neg\alpha)\vee\lambda,$$

equation (5.3) is satisfied by $\neg\alpha$, too.

Conversely, let $m \in \mathcal{M}$ and $\lambda \in \boldsymbol{B}$ and define the map $f : \boldsymbol{B} \rightarrow \boldsymbol{B}$ by setting $f(x) = x\vee\lambda$. Then, $f$ is order-preserving and $f(x\wedge y) = f(x)\wedge f(y)$. Then by Proposition 3.5, $m\vee\lambda$ is a complete theory whose degree of inconsistency is $\lambda$. □

To prove a completeness theorem, we have to prove at first some lemmas.

**Lemma 5.3.** *Let $\tau$ be a theory and $\alpha$ a formula. Then we can extend $\tau$ into a theory $\tau' \supseteq \tau$ in such a way that* :

i) $\tau'(\alpha) = \tau(\alpha)$ *and* $\tau'(\neg\alpha) = \tau(\alpha)\rightarrow\tau(\neg\alpha)$;

ii) $\alpha$ *is decidable in* $\tau'$;

iii) $Inc(\tau') = Inc(\tau)$.

*Proof.* Define $v$ by setting $v(x) = \tau(x)$ if $x \neq \neg\alpha$ and $v(\neg\alpha) = \tau(\alpha)\rightarrow\tau(\neg\alpha)$, and set $\tau' = \mathcal{D}(v)$. Then $\tau' \supseteq v \supseteq \tau$. To prove i) since $\alpha_1,\ldots,\alpha_n, \neg\alpha \vdash \alpha$ if and only if $\alpha_1,\ldots,\alpha_n \vdash \alpha$,

$$\tau'(\alpha) = Sup\{v(\alpha_1)\wedge\ldots\wedge v(\alpha_n) : \alpha_1,\ldots,\alpha_n \vdash \alpha \text{, where } \alpha_i \neq \neg\alpha\}$$
$$= Sup\{\tau(\alpha_1)\wedge\ldots\wedge\tau(\alpha_n) : \alpha_1,\ldots,\alpha_n \vdash \alpha\} = \tau(\alpha).$$

Also,

$$\tau'(\neg\alpha) = Sup\{v(\alpha_1)\wedge\ldots\wedge v(\alpha_{n+1}) : \alpha_1,\ldots,\alpha_n \vdash \neg\alpha\}$$
$$= v(\neg\alpha)\vee Sup\{v(\alpha_1)\wedge\ldots\wedge v(\alpha_n) : \alpha_i \neq \neg\alpha \text{ and } \alpha_1,\ldots,\alpha_n \vdash \neg\alpha\}$$

$= v(\neg\alpha)\vee Sup\{\tau(\alpha_1)\wedge...\wedge\tau(\alpha_n) : \alpha_i \neq \neg\,\alpha$ and $\alpha_1,...,\alpha_n \vdash \neg\alpha\}$
$\leq v(\neg\alpha)\vee Sup\{\tau(\alpha_1)\wedge...\wedge\tau(\alpha_n) : \alpha_1,...,\alpha_n \vdash \neg\alpha\}$
$= v(\neg\alpha)\vee\tau(\neg\alpha) = (\tau(\alpha)\rightarrow\tau(\neg\alpha))\vee\tau(\neg\alpha) = \tau(\alpha)\rightarrow\tau(\neg\alpha)$

To prove ii) observe that

$$\tau'(\alpha)\vee\tau'(\neg\alpha) = \tau(\alpha)\vee(\tau(\alpha)\rightarrow\tau(\neg\alpha)) = 1.$$

Finally, to prove iii) observe that

$$Inc(\tau') = \tau'(\alpha)\wedge\tau'(\neg\alpha) = \tau(\alpha)\wedge(\tau(\alpha)\rightarrow\tau(\neg\alpha)) = \tau(\alpha)\wedge\tau(\neg\alpha) = Inc(\tau).$$ □

**Lemma 5.4.** *Let $\tau$ be a theory and $\tau'$ a theory extending $\tau$ such that $Inc(\tau) = Inc(\tau')$. Then $\tau'(\beta) = \tau(\beta)$ for every formula $\beta$ decidable in $\tau$.*

*Proof.* Since $\tau(\beta)\vee\tau(\neg\beta) = 1$, we have that, $(\tau(\beta)\vee\tau(\neg\beta))\wedge -\tau(\neg\beta) = -\tau(\neg\beta)$ and therefore $\tau(\beta)\wedge -\tau(\neg\beta) = -\tau(\neg\beta)$. Consequently,

$$\tau(\beta) = (\tau(\beta)\wedge -\tau(\neg\beta))\vee(\tau(\beta)\wedge\tau(\neg\beta)) = (-\tau(\neg\beta)\vee(\tau(\beta))\wedge\tau(\neg\beta)) = -\tau(\neg\beta)\vee Inc(\tau).$$

Likewise, one proves that $\tau'(\beta) = -\tau'(\neg\beta)\vee Inc(\tau')$. Thus, since $-\tau(\neg\beta) \geq -\tau'(\neg\beta)$, and $Inc(\tau) = Inc(\tau')$, we have that

$$\tau(\beta) = -\tau(\neg\beta)\vee Inc(\tau) \geq -\tau'(\neg\beta)\vee Inc(\tau') = \tau'(\beta).$$

This proves that $\tau(\beta) = \tau'(\beta)$. □

**Lemma 5.5.** *Let $\tau$ be a theory and $\alpha$ a formula. Then we can extend $\tau$ into a complete theory $\tau_0$ such that $\tau_0(\alpha) = \tau(\alpha)$, and $Inc(\tau_0) = Inc(\tau)$. Consequently, any theory whose inconsistency degree is $\lambda$ is intersection of elements in $\mathcal{M}_\lambda$.*

*Proof.* Set

$$T = \{\tau' : \tau' \text{ is a theory}, \tau'\supseteq\tau, \tau'(\alpha) = \tau(\alpha), \alpha \text{ is decidable in } \tau', Inc(\tau') = Inc(\tau)\}.$$

Then, by Lemma 5.3, $T \neq \emptyset$. We claim that $T$ is inductive and therefore that a maximal element in $T$ exists. Indeed, let $(\tau_n)_{n\in N}$ be an order-preserving sequence of elements of $T$ and set $\tau' = \cup_{n\in N}\tau_n$. Then, since $\mathcal{D}(\cup_{n\in N}\tau_n) = \cup_{n\in N}\mathcal{D}(\tau_n) = \cup_{n\in N}\tau_n$, $\tau'$ is a theory. Since, it is immediate that $\tau'(\alpha) = \tau(\alpha)$, $\alpha$ is decidable in $\tau'$ and $Inc(\tau') = Inc(\tau)$, we can conclude that $\tau' \in T$.

Let $\tau_0$ be a maximal element in $T$, we have to prove that $\tau_0$ is complete. Indeed, otherwise a formula $\beta$ exists such that $\tau_0(\beta)\vee\tau_0(\neg\beta) < 1$. Let $\tau'$ be the extension of $\tau_0$ associated with $\beta$ we obtain by Lemma 5.3. Then, by Lemma 5.4, $\tau'(\alpha) = \tau_0(\alpha) = \tau(\alpha)$. Moreover, since $\tau'(\beta)\vee\tau'(\neg\beta) = 1$, $\tau'$ is a proper extension of $\tau_0$ belonging to $T$, a contradiction. □

As an immediate consequence of Lemma 5.5, we obtain the following *"completeness theorem"*.

**Theorem 5.6.** *Let **B** be a complete Boolean algebra. Then the logical consequence operator defined by the fuzzy semantics $\mathcal{M}^*$ coincides with the deduction operator of the $\wedge$-extension of the classical propositional calculus.*

Now we are able to give our main definition.

**Definition 5.6.** Let ***B*** be a complete Boolean algebra. Then we call *Boolean logic* in ***B*** the fuzzy logic defined by the $\wedge$-extension of the classical propositional calculus and by the fuzzy semantics $\mathcal{M}^*$.

Note that the expression *"Boolean logic"* has a different meaning in literature. This is due to the fact that in multivalued logic the deduction apparatus is usually devoted to generate the set of *tautologies* i.e. formulas whose valuations assume only designated values. This point of view was inherited from classical logic where, due to the compactness and the deduction theorems, the notion of deduction under hypotheses can be derived from the notion of tautology. On the other hand, the tautologies in a Boolean truth-functional semantics coincide with the tautologies in the classical propositional calculus. So, the deduction apparatus for classical logic fits well also for such a semantics. Totally different is the point of view of fuzzy logic we embrace. A logical apparatus is a tool to elaborate the available information and not merely a tool to produce tautologies. Also, usually this information is incomplete and partial inconsistent. In the case of Boolean information, this leads to Definition 5.6.

**6. Partially inconsistent belief measures and capacities.**

The proposed Boolean logic is useful to manage information, probabilistic in nature, in which both incompleteness and inconsistency are admitted. To show this, we will extend the basic notions of belief measure and capacity as defined in [12] and [5], respectively. Let $\boldsymbol{B}$ a complete Boolean algebra whose elements we call *events*, we define a *mass* as any map $m : \boldsymbol{B} \to [0,1]$ such that

j) $\sum_{e\in \boldsymbol{B}} m(e) = 1$.

jj) $m(0) = 0$.

The *belief measure* associated with a mass $m$ is the map $\mu : \boldsymbol{B} \to [0,1]$ defined by setting, for any $x\in \boldsymbol{B}$,

$$\mu(x) = \sum_{e\subseteq X} m(e). \tag{6.1}$$

We call *focal* the events $e$ such that $m(e) \neq 0$. Also, a map $\mu : \boldsymbol{B} \to [0,1]$ is a *capacity* if

i) $\mu(1) = 1$ ;

ii) for any integer $n$ and every $n$-tuple of events $x_1, \dots ,x_n$

$\mu(x_1\vee\ldots\vee x_n) \geq \sum\{(-1)^{|I|+1} \mu(x_{i(1)}\wedge\ldots\wedge x_{i(k)}) : I =\{i(1),\ldots, i(k)\} \subseteq \{1,\ldots, n\}\}$.

iii) $\mu(0) = 0$.

Every finitely additive probability is a capacity. One proves that, in the case $\boldsymbol{B}$ finite, $\mu$ is a capacity iff $\mu$ is a belief function, i.e. a mass $m$ exists such that (6.1) holds. Now, the conditions $m(0) = 0$ and $\mu(0) = 0$ are consistency conditions, in a sense. The validity of these conditions was discussed by many authors mainly in connection with the processes of fusion of information arising from different sources. For example, recall Dempster's rule of combination for belief measures and assume that the masses $m_1$ and $m_2$ define the beliefs $\mu_1$ and $\mu_2$, respectively. Then a new mass $m$ is defined by setting $m(0) = 0$ and, in the case $e \neq 0$,

$$m(e) = \frac{1}{1-k}\left(\sum\{m_1(x)\cdot m_2(x) : x\wedge y = e\}\right) \tag{6.2}$$

where $k = \sum \{m_1(x)\cdot m_2(y) : x\wedge y = 0\}$. We call the *orthogonal sum* of $\mu_1$ and $\mu_2$ the belief $\mu_1\oplus\mu_2$ associated with the mass $m$. Obviously, the composition is defined only if $k \neq 1$. We interpret $k$ as the *degree of inconsistency* between $\mu_1$ and $\mu_2$ (in [12] the quantity $Con(\mu_1,\mu_2) = log(1/(1-k))$ is called the *weight of conflict between* $\mu_1$ and $\mu_2$). Consequently, Dempster's rule is the result of a two-step process:

1. a fusion of the information arising from two different sources and represented by the quantities

$$m'(e) = \sum\{m_1(x)\cdot m_2(y) : x\wedge y = e\}, \tag{6.3}$$

2. a revision of the resulting information, obtained by setting $m(0) = 0$ in order to restate the consistency and, consequently, the normalization

$$m(e) = m'(e)/(1-k). \tag{6.4}$$

to obtain the condition $\sum_{e\in \boldsymbol{B}} m(e) = 1$.

Now, such a revision process in defining the orthogonal sum is a little unsatisfactory since it destroys part of the knowledge. Indeed, no trace of the partial conflict between the belief measures $\mu_1$ and $\mu_2$ remains in the sum $\mu_1\oplus\mu_2$. Moreover the normalization gives rise to some computational difficulties. For example, Shafer in [12] claims that: *"I treated the normalizing constant ... as a nuisance. And it does complicate the description of Dempster's rule and the calculation of the orthogonal sums"*. In accordance with these considerations, we will admit partial inconsistency and therefore to skip out condition jj) in defining the belief measures and condition iii) in defining the capacities (see also [13]).

**Definition 6.1.** A map $m : \boldsymbol{B}\to [0,1]$ is a *generalized mass* if j) is satisfied. A *generalized belief measure* is a map $\mu : \boldsymbol{B} \to [0,1]$ defined from a generalized mass $m$ by (6.1).

Such a definition enables us to define the *generalized orthogonal sum* of two generalized belief measures with masses $m_1$ and $m_2$ as the generalized belief measure whose generalized mass $m'$ is defined by (6.3).

**Definition 6.2.** A map $\mu : \boldsymbol{B} \to [0,1]$ is a *generalized capacity* if i) and ii) are satisfied. We say that $\mu$ is a *generalized probability* if $\mu(1) = 1$ and

$$\mu(x \vee y) + \mu(x \wedge y) = \mu(x) + \mu(y).$$

We call *degree of inconsistency* of a generalized belief measure (capacity) $\mu$ the value $Inc(\mu) = \mu(0)$. Then, the capacities (probabilities) are the generalized capacities (probabilities) whose inconsistency degree is 0. The map $\mu$ constantly equal to 1 is a generalized probability such that $Inc(\mu) = 1$.

The proof of the following proposition is trivial:

**Proposition 6.3.** *Assume that* $\boldsymbol{B}$ *is finite. Then the generalized capacities coincide with the generalized belief measures.*

In order to allow a logical treatment of the generalized belief measures we have to adjust the proposed notions by considering functions defined in the set $\mathbb{F}$ of formulas. So, we say that $p : \mathbb{F} \to [0,1]$ is a *generalized capacity* in $\mathbb{F}$ if

i) $p$ is compatible with the logical equivalence,
ii) $\alpha$ tautology $\Rightarrow p(\alpha) = 1$
iii) for every $n$-tuple of formulas $\alpha_1, \ldots, \alpha_n$

$$p(\alpha_1 \vee \ldots \vee \alpha_n) \geq \sum\{(-1)^{|I|+1} p(\alpha_{i(1)} \wedge \ldots \wedge \alpha_{i(k)}) : I = \{i(1), \ldots, i(k)\} \subseteq \{1,\ldots,n\}\}.$$

If instead of iii) the equation

iii') $p(\alpha \vee \beta) + p(\alpha \wedge \beta) = p(\alpha) + p(\beta)$

is satisfied, then we say that $p$ is a *generalized probability* in $\mathbb{F}$. A generalized capacity $p$ such that

iv) $\phi$ contradiction $\Rightarrow p(\phi) = 0$,

is called a *capacity* in $\mathbb{F}$. A generalized probability satisfying iv) is called a (*finitely additive*) *probability* in $\mathbb{F}$.

**Proposition 6.4.** *Let* $p: \mathbb{F} \to [0,1]$ *be a generalized capacity. Then p is a capacity iff, for every* $\alpha \in \mathbb{F}$,

$$p(\alpha) + p(\neg\alpha) \leq 1, \tag{6.5}$$

*p is a generalized probability iff, for every* $\alpha \in \mathbb{F}$,

$$p(\alpha) + p(\neg\alpha) = 1 + Inc(p), \tag{6.6}$$

*p is a finitely additive probability iff, for every* $\alpha \in \mathbb{F}$,

$$p(\alpha) + p(\neg\alpha) = 1. \tag{6.7}$$

*Proof.* Assume that $p$ satisfies (6.5) and that $\phi$ is a contradiction. Then $\neg\phi$ is a tautology and therefore $p(\neg\phi) = 1$. Consequently, by (6.5), we have that $p(\phi)+p(\neg\phi) = p(\phi)+1 \leq 1$. This proves that $p(\phi) = 0$ and therefore that $p$ is a capacity. Conversely, assume that $p$ is a capacity. Then for every formula $\alpha$, $p(\alpha \wedge \neg\alpha) = 0$ and therefore

$$1 = p(\alpha \vee \neg\alpha) \geq p(\alpha) + p(\neg\alpha) - p(\alpha \wedge \neg\alpha) = p(\alpha) + p(\neg\alpha).$$

Assume that $p$ is a capacity satisfying (6.6). Then, by observing that $p(-\alpha \wedge -\beta) \geq -p(-\alpha \vee -\beta) + p(-\alpha) + p(-\beta)$, we have that

$$\begin{aligned}
p(\alpha \vee \beta) + p(\alpha \wedge \beta) &= p(-(-\alpha \wedge -\beta)) + p(\alpha \wedge \beta) = 1 + Inc(p) - p(-\alpha \wedge -\beta) + p(\alpha \wedge \beta)\\
&\leq 1 + Inc(p) - [-p(-\alpha \vee -\beta) + p(-\alpha) + p(-\beta)] + p(\alpha \wedge \beta)\\
&= 1 + Inc(p) - [-p(-(\alpha \wedge \beta)) + 1 + Inc(p) - p(\alpha) + 1 + Inc(p) - p(\beta)] + p(\alpha \wedge \beta)\\
&= 1 + Inc(p) + (1 + Inc(p) - p(\alpha \wedge \beta)) - 1 - Inc(p) + p(\alpha) - 1 - Inc(p) + p(\beta)) + p(\alpha \wedge \beta)\\
&= p(\alpha) + p(\beta).
\end{aligned}$$

Since it is also $p(\alpha \vee \beta) + p(\alpha \wedge \beta) \geq p(\alpha) + p(\beta)$, we can conclude that $p$ is a generalized probability. Conversely, it is obvious that any generalized probability is a generalized capacity satisfying (6.6).

Assume that $p$ is a generalized capacity satisfying (6.7). Then, given two formulas $\alpha$ and $\beta$,

$$p(\neg\alpha \wedge \neg\beta) + p(\neg\alpha \vee \neg\beta) \geq p(\neg\alpha) + p(\neg\beta)$$

and therefore, since $\neg\alpha \wedge \neg\beta \equiv \neg(\alpha \vee \beta)$ and $\neg\alpha \vee \neg\beta \equiv \neg(\alpha \wedge \beta)$,

$$1 - p(\alpha \vee \beta) + 1 - p(\alpha \wedge \beta) = p(\neg(\alpha \vee \beta)) + p(\neg(\alpha \wedge \beta)) \geq 1 - p(\alpha) + 1 - p(\beta).$$

i.e.,

$$p(\alpha\vee\beta) + p(\alpha\wedge\beta) \leq p(\alpha) + p(\alpha).$$

This proves that $p$ is a generalized probability. Since

$$1+Inc(p) = p(\alpha\vee\neg\alpha)+p(\alpha\wedge\neg\alpha) = p(\alpha)+p(\neg\alpha) = 1,$$

$Inc(p) = 0$ and $p$ is a finitely additive probability.

Trivially, any finitely additive probability is a generalized capacity satisfying (6.7). □

**7. Boolean extensions and generalized belief measures.**

Let $\tau$ be a theory and $\mu : \boldsymbol{B} \to [0,1]$ a map. Then we define a valuation $\mu_\tau : \mathbb{F} \to [0,1]$ of the formulas by setting, for any $\alpha\in\mathbb{F}$,

$$\mu_\tau(\alpha) = \mu(\tau(\alpha)). \tag{7.1}$$

Such a kind of functions were considered by A. Saffiotti in [11] under the hypotheses that $\tau$ is a truth-functional valuation and $\mu$ a capacity.

**Theorem 7.1.** *Let* $\mu : \boldsymbol{B} \to [0,1]$ *be a generalized capacity and* $\tau$ *a theory. Then the map* $\mu_\tau$ *defined by* (7.1) *is a generalized capacity such that* $Inc(\mu_\tau) = \mu(Inc(\tau))$. *In particular, if* $\mu$ *is a capacity and* $\tau$ *is totally consistent, then* $\mu_\tau$ *is a capacity.*

*Proof.* We have, by iv) of Theorem 3.2, that $\tau(\alpha_1\vee\ldots\vee\alpha_n) \supseteq \cup_{i=1,\ldots,n}\tau(\alpha_i)$. Consequently, by iii) of Theorem 4.1,

$$\begin{aligned}\mu_\tau(\alpha_1\vee \ldots \vee\alpha_n) &= \mu(\tau(\alpha_1\vee\ldots\vee\alpha_n)) \geq \mu(\cup_{i=1,\ldots,n}\tau(\alpha_i))\\ &\geq \sum\{(-1)^{|I|+1}\mu(\tau(\alpha_{i(1)})\wedge\ldots\wedge\tau(\alpha_{i(k)})) : I = \{i(1), \ldots, i(k)\}\subseteq\{1, \ldots, n\}\}\\ &= \sum\{(-1)^{|I|+1}\mu(\tau(\alpha_{i(1)}\wedge\ldots\wedge\alpha_{i(k)})) : I = \{i(1), \ldots, i(k)\}\subseteq\{1, \ldots, n\}\}\\ &= \sum\{(-1)^{|I|+1}\mu_\tau(\alpha_{i(1)}\wedge\ldots\wedge\alpha_{i(k)}) : I = \{i(1), \ldots, i(k)\}\subseteq\{1, \ldots, n\}\}.\end{aligned}$$

Then $p_\tau$ is a generalized capacity. It is obvious that $Inc(\mu_\tau) = \mu(Inc(\tau))$. □

**Theorem 7.2.** *Let* $\mu$ *be a probability and* $\tau$ *a complete theory. Then the map* $\mu_\tau$ *defined by* (7.1) *is a generalized probability. If* $\tau$ *is complete and totally consistent, then* $\mu_\tau$ *is a probability.*

*Proof.* Assume that $\tau$ is complete. Then, by Theorem 4.6, $\tau(\alpha\vee\beta) = \tau(\alpha)\vee\tau(\beta)$. Consequently,

$$\begin{aligned}p_\tau(\alpha\vee\beta) + p_\tau(\alpha\wedge\beta) &= \mu(\tau(\alpha\vee\beta)) + \mu(\tau(\alpha\wedge\beta)) = \mu(\tau(\alpha)\vee\tau(\beta)) + \mu(\tau(\alpha)\wedge\tau(\beta))\\ &= \mu(\tau(\alpha)) + \mu(\tau(\beta)) = p_\tau(\alpha) + p_\tau(\beta).\end{aligned}$$

This proves that $p_\tau$ is a generalized probability. It is obvious that if $\tau$ is totally consistent, then $Inc(p_\tau) = 0$ and therefore $p_\tau$ is a probability. □

**Theorem 7.3.** *Assume that* $\mu$ *is a finitely additive probability such that* $\mu(x) = 0$ *only if* $x = 0$ *and let* $\tau$ *be a theory. Then,*

*i)* $\mu_\tau$ *is a generalized probability iff* $\tau$ *is complete.*

ii) $\mu_\tau$ *is a capacity iff* $\tau$ *is totally consistent.*

iii) $\mu_\tau$ *is a finitely additive probability iff* $\tau$ *is complete and totally consistent.*

*Proof.* Firstly, observe that $\mu(y) = 1$ entails that $y = 1$. Indeed, if $\mu(y) = 1$, then

$$1 = \mu((y\vee(-y)) = \mu(y)+\mu(-y) = 1+\mu(-y)$$

and therefore $\mu(-y) = 0$. This entails that $-y = 0$ and therefore that $y = 1$.

i) Assume that $\mu_\tau$ is a generalized probability. Then, by Proposition 6.4,

$$\mu_\tau(\alpha)+\mu_\tau(\neg\alpha) = 1+Inc(\mu_\tau) = 1+\mu(Inc(\tau)),$$

and, by Theorem 4.1,

$$\begin{aligned}\mu_\tau(\alpha)+\mu_\tau(\neg\alpha) &= \mu(\tau(\alpha))+\mu(\tau(\neg\alpha)) = \mu(\tau(\alpha)\vee\tau(\neg\alpha)))+\mu(\tau(\alpha)\wedge\tau(\neg\alpha)))\\ &= \mu(\tau(\alpha)\vee\tau(\neg\alpha))+\mu(Inc(\tau)).\end{aligned}$$

Then $\mu(\tau(\alpha)\vee\tau(\neg\alpha))) = 1$ and therefore $\tau(\alpha)\vee\tau(\neg\alpha)) = 1$ and this proves that $\tau$ is complete. Assume that $\tau$ is complete. Then, by Theorem 7.2, $\mu_\tau$ is a generalized probability.

ii) Assume that $\mu_\tau$ is a capacity. Then, since the degree of inconsistency $\mu(Inc(\tau))$ of $\mu_\tau$ is equal to zero, and since $\mu(x) = 0$ only if $x = 0$ we have that $Inc(\tau) = \tau(\alpha\wedge\neg\alpha) = 0$. This proves that $\tau$ is totally consistent. The converse implication follows from Theorem 7.1.
iii) An immediate consequence of i) and ii). □

By Proposition 6.4, we can reformulate Theorem 7.3 as follows:

**Theorem 7.4.** *Assume that $\mu$ is a finitely additive probability such that $\mu(x) = 0$ only if $x = 0$ and that $\tau$ is a theory. Then,*
i) *$\tau$ is complete iff, for any formula $\alpha$,*

$$\mu_\tau(\alpha)+ \mu_\tau(\neg\alpha) = 1+ Inc(\mu_\tau).$$

ii) *$\tau$ is totally consistent iff, for any formula $\alpha$,*

$$\mu_\tau(\alpha)+ \mu_\tau(\neg\alpha) \leq 1.$$

iii) *$\tau$ is complete and totally consistent iff, for any formula $\alpha$,*

$$\mu_\tau(\alpha)+ \mu_\tau(\neg\alpha) = 1.$$

Given a probability $\mu : \boldsymbol{B}\to[0,1]$, we can define an inferential apparatus, probabilistic in nature, by the operator $H : \boldsymbol{B}^{\mathbb{F}} \to [0,1]^{\mathbb{F}}$ defined by setting, for any $v\in \boldsymbol{B}^{\mathbb{F}}$

$$H(v)(\alpha) = \mu(\mathcal{D}^{\cap}(v)(\alpha)). \tag{7.2}$$

In other words, our belief degree $H(v)(\alpha)$ on $\alpha$ is equal to the frequency of the cases in which $\alpha$ can be proved where the frequency is referred to the Boolean information $v$. The basic idea is that a deduction apparatus in a probabilistic setting has to elaborate Boolean information (by the operator $\mathcal{D}^{\cap}$) and not numerical information. Obviously, it is possible that $Inc(H(v)) \neq 0$. In such a case the value $H(v)(\alpha)$ does not give information on the actual degree of validity of a formula $\alpha$. Indeed, even if we are able to prove $\alpha$ to a degree $H(v)(\alpha)$ different from zero, in the case $H(v)(\alpha) = Inc(v)$ no reason exists for the validity of $\alpha$. We may affirm that $v$ confirms $\alpha$ only if $H(v)(\alpha) > Inc(H(v))$. Accordingly, the logic represented by $H$ is not monotone in spite of the fact that $H$ is a monotone operator. As a matter of fact, the degree of validity of a formula $\alpha$ is represented by the difference $H(v)(\alpha)$-$Inc(H(v))$ rather than by $H(v)(\alpha)$. On the other hand, since both $H(v)(\alpha)$ and $Inc(H(v))$ are increasing functions of $v$, their difference is not an increasing function of $v$, in general.

Owing to the above consideration, perhaps it would be better to substitute the operator $H$ with a *"normalized"* operator (see also [8], Section 9 of Chapter 2 and Section 4 of Chapter 6). To this aim we prove at first the following proposition.

**Proposition 7.5.** *Let $p : \mathbb{F} \to [0,1]$ be a generalized capacity such that $Inc(p) \neq 1$ and define the operator $N(p) : \mathbb{F}\to [0,1]$ by setting, for any $\alpha\in\mathbb{F}$,*

$$N(p)(\alpha) = \frac{p(\alpha) - Inc(p)}{1 - Inc(p)} \tag{7.3}$$

*Then $N(p)$ is a capacity. If $p$ is a generalized probability, then $N(p)$ is a probability.*

*Proof.* It is evident that, for any tautology $\alpha$, $N(p)(\alpha) = 1$. Moreover, since by hypothesis for any integer $n$ and every $n$-tuple of formulas $\alpha_1, \dots ,\alpha_n$,

$$p(\alpha_1\vee\ldots\vee\alpha_n) \geq \Sigma\{(-1)^{|I|+1} p(\alpha_{i(1)}\wedge\ldots\wedge\alpha_{i(k)}) : I =\{i(1),\ldots, i(k)\} \subseteq \{1,\ldots, n\}\}.$$

and since

$$\Sigma\{(-1)^{|I|+1} : I = \{i(1),\ldots, i(k)\} \subseteq \{1,\ldots, n\}\} = 1$$

we have that

$$p(\alpha_1\vee\ldots\vee\alpha_n) - Inc(p) \geq \Sigma\{(-1)^{|I|+1} p(\alpha_{i(1)}\wedge\ldots\wedge\alpha_{i(k)}) : I =\{i(1),\ldots, i(k)\}\subseteq\{1,\ldots, n\}\} - Inc(p)$$
$$= \Sigma\{(-1)^{|I|+1}[p(\alpha_{i(1)}\wedge\ldots\wedge\alpha_{i(k)}) - Inc(p)] : I =\{i(1),\ldots, i(k)\}\subseteq\{1,\ldots, n\}\}.$$

By dividing by 1-$Inc(p)$ both the sides of such an inequality, we prove that $N(p)$ satisfies ii) and therefore that $N(p)$ is a generalized capacity. Since it is obvious that $Inc(N(p)) = 0$, $N(p)$ is a capacity. The remaining part of the proposition is trivial. □

Observe that if $p$ is the generalized belief measure resulting by the generalized orthogonal sum of two belief measures, then by (7.3) we obtain the usual orthogonal sum of these measures.

In accordance with Proposition 7.5, we can define a non-monotone inferential apparatus by the *normalized* operator $H_n : \boldsymbol{B}^{\mathbb{F}} \to [0,1]^{\mathbb{F}}$ defined by setting, for any $v \in \boldsymbol{B}^{\mathbb{F}}$

$$H_n(v)(\alpha) = \frac{\mu(\mathcal{D}^{\cap}(v)(\alpha)) - \mu(Inc(v))}{1 - \mu(Inc(v))} \tag{7.4}$$

**Theorem 7.6.** Given an initial valuation $v : \mathbb{F} \to \boldsymbol{B}$, the mapping $H_n(v) : \mathbb{F} \to [0,1]$ is a capacity. If $v$ is complete, then $H_n(v)$ is a probability.

*Proof.* Trivial. □

**8. Possible worlds semantics and Boolean extensions**

Let $W$ be a nonempty set whose elements we call *worlds* or *past cases* and consider the Boolean algebra $\boldsymbol{B} = P(W)$. Given an initial valuation $v : \mathbb{F} \to \boldsymbol{B}$, for any formula $\alpha$ we interpret $v(\alpha)$ as the set of worlds in which we know that $\alpha$ is true (equivalently, the set of past cases in which we know that $\alpha$ was true). In the resulting canonical extension the $\boldsymbol{B}$-subset $a : \mathbb{F} \to \boldsymbol{B}$ of logical axioms is defined by setting $a(\alpha) = W$ if $\alpha$ is a tautology and $a(\alpha) = \emptyset$ otherwise. The canonical extension of the classical Modus Ponens, is pictured by

$$\frac{\alpha\,,\,\alpha \to \beta}{\beta} \qquad\qquad \frac{X\,,\,Y}{X \cap Y}\;,$$

and it claims that if we know that $\alpha$ is satisfied in the set $X$ of worlds and $\alpha \to \beta$ in the set $Y$ of worlds, then we can say that $\beta$ holds in the set $X \cap Y$ of worlds. Moreover, in accordance with the results exposed in Section 3, we have that:

$$\mathcal{D}^{\cap}(v)(\alpha) = \begin{cases} W & \textit{if } \alpha \textit{ is a tautology} \\ \bigcup\{v(\alpha_1) \cap \ldots \cap v(\alpha_n) : \alpha_1,\ldots,\alpha_n \vdash \alpha\} & \textit{otherwise.} \end{cases} \tag{8.1}$$

Then, while $v(\alpha)$ is the set of the worlds in which <u>we know</u> that $\alpha$ is true, $\mathcal{D}^{\cap}(v)(\alpha)$ represents the set of worlds in which <u>we can prove</u> that $\alpha$ is true given the information $v$. In accordance, since for any contradiction $\phi$, $Inc(v) = \mathcal{D}^{\cap}(v)(\phi)$, we interpret $Inc(v)$ as the set of world in which the information is inconsistent.

It is interesting to examine the meaning of the cuts. Indeed, given a set $X$ of worlds, the cut $C(v,X)$ of an initial valuation $v$ is the $\{\alpha \in \mathbb{F} : v(\alpha) \supseteq X\}$ of formulas that we know to be true in any world in $X$. In particular, we are interested in the *singleton-cuts* $C(v,c) = C(v,\{c\})$, where $c \in W$. $C(v,c)$ is the information we have on $c$. Since, given any subset $X$ of $W$,

$$C(v,X) = \cap_{c \in X} C(v,c),$$

any cut can be defined by singleton cuts. Also, it is obvious that

$$v(\alpha) = \{c \in W : \alpha \in C(v,c)\}. \tag{8.2}$$

The singleton cuts enable us to give a different way to represent the Boolean algebra $\boldsymbol{B}^{\mathbb{F}}$. Indeed, we consider the direct power $P(\mathbb{F})^W$ of the algebra $P(\mathbb{F})$ with index set $W$. The following proposition shows that $\boldsymbol{B}^{\mathbb{F}}$ is isomorphic to $P(\mathbb{F})^W$.

**Proposition 8.1.** *Let* $h : \boldsymbol{B}^{\mathbb{F}} \to P(\mathbb{F})^W$ *be defined by setting, for any* $v \in \boldsymbol{B}^{\mathbb{F}}$,

$$h(v) = (C(v,c))_{c \in W} \tag{8.3}$$

*Then h is an isomorphism between the Boolean algebras* $\boldsymbol{B}^{\mathbb{F}}$ *and* $P(\mathbb{F})^W$. *The inverse* $h^{-1}$ *associates any* $(T_c)_{c \in W}$ *in* $P(\mathbb{F})^W$ *with the* $\boldsymbol{B}$*-subset v defined by setting*

$$v(\alpha) = \{c \in W : \alpha \in T_c\}. \tag{8.4}$$

*Proof.* Let $s_1$ and $s_2$ be two ***B***-subsets. Then, since

$$C(s_1 \cup s_2, c) = C(s_1,c) \cup C(s_2,c),\ C(s_1 \cap s_2, c) = C(s_1,c) \cap C(s_2,c),$$

we have that $h(s_1 \cup s_2) = h(s_1) \cup h(s_2)$ and $h(s_1 \cap s_2) = h(s_1) \cap h(s_2)$. Moreover, since $C(-s_1, c) = -C(s_1,c)$, it is $h(-s_1) = -h(s_1)$ and this proves that $h$ is a homomorphism. Assume that $h(s_1) = h(s_2)$ and therefore that, for any $c \in W$, $C(s_1,c) = C(s_2,c)$. Then, by (8.2), $s_1 = s_2$. This proves that $h$ is injective. Finally, let $(T_c)_{c \in W}$ be an element in $P(\mathbb{F})^W$, and define $v$ as in (8.4). Then, since,

$$\alpha \in C(v,c) \iff c \in v(\alpha) \iff \alpha \in T_c,$$

we have that $C(v,c) = T_c$ and therefore that $h(v) = (T_c)_{c \in W}$. This proves that $h$ is surjective. □

The next theorem shows that the following diagram commutes:

$$\begin{array}{ccc} v & \xrightarrow{\ h\ } & (C(v,c))_{c \in W} \\ \downarrow & & \downarrow \\ \mathcal{D}^{\frown}(v) & \xleftarrow{\ h^{-1}\ } & (\mathcal{D}(C(v,c)))_{c \in W} \end{array}$$

**Theorem 8.2.** *Given an initial valuation $v$ and $c \in W$,*

$$C(\mathcal{D}^{\frown}(v), c) = \mathcal{D}(C(v,c)). \qquad (8.5)$$

*Consequently,*

$$\mathcal{D}^{\frown}(v)(\alpha) = \{c \in W : C(v,c) \vdash \alpha\}. \qquad (8.6)$$

*Proof.* If $\alpha$ is a tautology, then we have that $\alpha \in C(\mathcal{D}^{\frown}(v),c)$ and $\alpha \in \mathcal{D}(C(v,c))$. Otherwise, we have that, for any formula $\alpha$,

$\alpha \in C(\mathcal{D}^{\frown}(v),c) \iff c \in \mathcal{D}^{\frown}(v)(\alpha)$

$\iff$ there are $\alpha_1,...,\alpha_n$ such that $c \in v(\alpha_1),..., c \in v(\alpha_n)$ and $\alpha_1,...,\alpha_n \vdash \alpha$

$\iff$ there are $\alpha_1 \in C(v,c),...,\alpha_n \in C(v,c)$ such that $\alpha_1,...,\alpha_n \vdash \alpha$

$\iff C(v,c) \vdash \alpha \iff \alpha \in \mathcal{D}(C(v,c))$. □

In other words, the deductions in the canonical extension can be reduced to the deductions on the cuts. It is worth noticing that it is possible that our knowledge is not complete, i.e. that a formula $\alpha$ exists such that $\mathcal{D}^{\frown}(v)(\alpha) \cup \mathcal{D}^{\frown}(v)(\neg\alpha) \neq W$. This means that a world $c$ exists such that neither $\alpha$ or $\neg\alpha$ can be proved in $c$.

**Corollary 8.3.** *Let $\tau$ be a **B**-subset of formulas. Then, $\tau$ is a (complete, totally consistent) theory iff, for any $c \in W$, $C(\tau,c)$ is a (complete, consistent) theory.*

*Proof.* Assume that $\tau$ is a theory, then by Theorem 4.2 each $C(\tau,c)$ is a theory. Conversely, assume that each $C(\tau,c)$ is a theory and observe that, given any element $X \in$ ***B***, $C(\tau,X) = \bigcap\{C(\tau,c) : c \in X\}$. By recalling that the intersection of a family of theories is a theory, we have that $C(\tau,X)$ is a theory. By Theorem 4.2 this entails that $\tau$ is a theory.

Assume that $\tau$ is complete. Then, since $\tau(\alpha) \cup \tau(\neg\alpha) = W$, given any $c \in W$, either $c \in \tau(\alpha)$ or $c \in \tau(\neg\alpha)$, i.e. either $\alpha \in C(\tau,c)$ or $\neg\alpha \in C(\tau,c)$. This proves that $C(\tau,c)$ is complete. Conversely, assume that, for every $c \in W$, $C(\tau,c)$ is complete and therefore that, for any formula $\alpha$, either $\alpha \in C(\tau,c)$ or $\neg\alpha \in C(\tau,c)$. Then, for every $c \in W$, either $c \in \tau(\alpha)$ or $c \in \tau(\neg\alpha)$. This proves that $\tau(\alpha) \cup \tau(\neg\alpha) = W$ and therefore that $\tau$ is complete.

Finally, by Theorem 4.2, $\tau$ is totally consistent iff each $C(\tau,c)$ is consistent. □

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