Point-free geometry, Approximate Distances and Verisimilitude of Theories

by
Giangiacomo Gerla*

Department of Mathematics and Information Sciences, University of Salerno
Via Ponte Don Melillo 84084 Fisciano (SA) Italy

Abstract: We propose a metric approach to Popper's verisimilitude question which is related with point-free geometry. Indeed, we define the theory of *approximate metric spaces* whose primitive notion are *regions*, *inclusion relation*, *minimum distance*, and *maximum distance* between regions. Then, we show that the class of possible scientific theories has a structure of approximate metric space. So, we can define the verisimilitude of a theory as a function of its (approximate) distance from the truth. This avoids some of the difficulties arising from the known definitions of verisimilitude.

Keywords: Verisimilitude, Popper, point-free geometry, metric spaces, multi-valued logic.

1. Introduction

In his definition of the notion of verisimilitude, Popper [15] claims that a theory T_2 has to be considered *closer to the truth* than a theory T_1 if and only if

(*i*) all the true consequences of T_1 are consequences of T_2,

(*ii*) all the false consequences of T_2 are consequences of T_1,

(*iii*) either some true consequences of T_2 are not consequences of T_1 or some false consequences of T_1 are not consequences of T_2.

In other words, T_2 is able to prove all the theorems of T_1 which are in accordance with the evidence, the theorems of T_2 conflicting with the evidence are also theorems of T_1 and T_2 is an effective *"progress"* with respect to T_1. The philosophical motivation behind Popper's proposal was to solve the problem of scientific progress by showing that it is possible that false theories are closer to the truth than other false theories. Unfortunately, a fatal flaw in this definition was detected independently by Tichý [18], and Miller [10]. They showed that, by admitting Popper's definition, no false theory is closer to the truth than any other false theory. As a consequence, several different definitions were proposed to avoid these difficulties but, in my opinion, no one completely satisfactory.

This paper concerns a distance-based approach to the definition of verisimilitude in accordance with the point of view of Miller and other authors (see for example [8, 11, 12, 13]), for which *"if theories can be close to or distant from the truth, then presumably they can be close to or distant from one another"*. Namely, the framework we refer to is not the usual point-based metric space theory. We refer to "point-free geometry" whose basic idea is that regions have to be assumed as primitive and points have to be defined in some way by *"abstraction processes"* (see the deep analysis proposed by A. N. Whitehead in [19], [20] and [21]). More precisely, we define a class of structures in which, apart from the *regions* and the *inclusion* relation, the *minimum distance* and the *maximum distance* between regions are assumed as primitives. We call these structures *approximate metric spaces*. Successively, we equip the class of possible scientific theories with a structure of approximate metric space and we define the verisimilitude of a theory as a function of its approximate distance from the truth (see also the definition of verisimilitude proposed in [5]). The motivation of such a choice is that usually a scientific theory T is not complete. So, it is misleading to identify T with a point in a metric space. On the contrary T represents a set of possible points (i.e. *worlds*), and therefore a "region", in this space, in general. Only the complete theories can play the role of points.

The obtained definition faces several known paradoxes but it turns out to be inadequate in some respects. On the other hand, the same kind of inadequateness is shared by all the definitions we know and we have no pretension to give a definitive solution to the question of verisimilitude (perhaps an impossible enterprise). We will rather elaborate toy-models and some new mathematical frameworks as tools to go on in the analysis of this basic notion. Also, we hope that the proposed concepts will turn out to be useful for a metric approach to point-free geometry and to formal logic.

2. Some difficulties in defining verisimilitude.

In this section we recall some well known objections to the existing definitions of verisimilitude. In order to do this, we refer to the following elementary example. Suppose that the weather outside is *hot* or *cold*, *rainy* or *dry*, *windy* or *calm* and denote these possibilities by the symbols h, $\neg h$, r, $\neg r$, w and $\neg w$, respectively. The resulting language refers to a universe with 8 possible worlds described by the corresponding complete theories

w_1. $\{h, r, w\}$, w_2. $\{\neg h, r, w\}$ w_3. $\{h, \neg r, w\}$, w_4. $\{h, r, \neg w\}$,
w_5. $\{\neg h, \neg r, \neg w\}$, w_6. $\{h, \neg r, \neg w\}$, w_7. $\{\neg h, r, \neg w\}$, w_8. $\{\neg h, \neg r, w\}$.

Suppose that, as a matter of fact, it is hot, rainy, and windy, i.e. that the truth is represented by the theory $\mathbb{V} = \{h, r, w\}$ describing the actual world w_1. Now, imagine we face two competing theories about the weather:

$T_2 = \{\neg h, r, w\}$ claiming that it's cold, rainy and windy

and

$T_1 = \{\neg h, \neg r, \neg w\}$ claiming that it's cold, dry and calm.

Both theories are false, but intuitively, one might think that T_2 is closer to the truth than T_1. Unfortunately, Popper's definition does not yield that result. Indeed T_1 has true consequences, for example $\neg((\neg h)\wedge r\wedge w)$, that T_2 does not have and T_2 has false consequences, for example $(\neg h)\wedge r\wedge w$, that T_1 does not have.

We can reformulate Popper's qualitative definition with a definition that is quantitative in nature by saying that T_2 is closer to the truth than T_1 if T_2 has *"more truths"* and *"fewer falsehoods"* than T_1in its set of deductive consequences . In other words we compare T_2 and T_1 by comparing the *number* of true and false consequences of these theories. Obviously, we can give a meaning to this definition only by identifying two logically equivalent formulas. Again, this does not solve the problem since T_2 and T_1 have exactly the same numbers of true consequences. To show this observe that we can identify any sentence α with the set $w(\alpha)$ of worlds in $W = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\}$ satisfying α. Accordingly, α is a consequence of a theory T if the set $w(T)$ of worlds satisfying T is contained into $w(\alpha)$. In particular, the set of true consequences of T_2 coincides with the class of subsets of W containing w_1 and w_2 and the set of true consequence of T_1 with the class of subsets of W containing w_1 and w_5. By symmetry, these classes contain the same number of elements. The same happens for the false sentences of T_2 and T_1.

Following Tichy [18], we can also reformulate such a quantitative approach by referring only to the atomic formulas h, r and w. Then T_2 is closer to the truth than T_1 because it makes one mistake but gets two things right while T_1 is wrong on all the three counts. An objection is that such a definition is not *"language invariant"* since, as proved by Miller in [10], this ordering may be reversed under simple changes of language. Indeed, suppose a new language in which one says that the weather is *Minnesotan* (m) provided that it is either hot and rainy or cold and dry, and *Arizonan* (a) if and only if it is either hot and windy or cold and calm. We may now re-express all the theories according to whether they say the weather is hot, Minnesotan, and Arizonan. That is, we may replace the predicates $\{h, r, w\}$ with $\{h, m, a\}$ and retain exactly the same expressive power. The true theory is re-expressed as $\{h, m, a\}$, T_2 is re-expressed as $\{\neg h, \neg m, \neg a\}$, T_1 is re-expressed as $\{\neg h, m, a\}$. Then T_2 makes three mistakes and T_1 makes one mistake and the order is reversed !

Further difficulties concern the fact that the completeness of a significant (axiomatizable) theory about the nature of the world looks unlikely. Indeed, since any complete and axiomatizable theory is decidable, such a completeness would be a rather surprising fact. If this is true, then the "degree of completeness" or "informative content" of a theory must play a basic role in evaluating its progress toward the truth. As an example, the set of logically true sentences is a totally true theory but we cannot claim its closeness to the truth. In the weather example we can assume that T_2 is the theory generated by $h \vee r$ and T_1 the theory generated by $h \vee r \vee w$. Then T_2 looks to be a progress with respect to T_1 since T_2 gives more information than T_1. On the other hand, since no literal h, r, w, $\neg h$, $\neg r$, $\neg w$ can be proved in these theories, Tichy's criterion cannot be applied.

3. Distances between sets

We recall some elementary notions in the theory of metric spaces. Denote the set of real numbers (non negative real numbers) by $\mathbb{R}$ (by $\mathbb{R}^+$), then a *pseudo-metric space* is the structure (M, δ) where $\delta : M \times M \to \mathbb{R}^+$ satisfies

1. $\delta(x,x) = 0$
2. $\delta(x,y) = \delta(y,x)$
3. $\delta(x,y) \leq \delta(x,z) + \delta(z,y)$.

A *metric space* is a pseudo-metric space such that

4. $\delta(x,y) = 0 \Rightarrow x = y$.

Any pseudo-metric space (M, δ) is associated with a metric space $(\underline{M}, \underline{\delta})$ obtained:

- by considering the equivalence relation $\equiv$ defined by setting $x \equiv y$ provided that $\delta(x,y) = 0$,
- by setting $\underline{M}$ equal to the quotient $M/\equiv$,
- by defining the function $\underline{\delta} : \underline{M} \times \underline{M} \to \mathbb{R}^+$ given by the equality

$$\underline{\delta}([x],[y]) = \delta(x,y).$$

Consider a class R of bounded non empty subsets of M interpreted as the *class of regions* in M. We define two maps $d : R \times R \to \mathbb{R}^+$ and $D : R \times R \to \mathbb{R}^+$ by setting, for any X and Y in R,

$$d(X,Y) = inf\{\delta(x, y) : x \in X, \ y \in Y\} \tag{3.1}$$

and

$$D(X,Y) = sup\{\delta(x, y) : x \in X, \ y \in Y\}. \tag{3.2}$$

The number $d(X,Y)$ is called the *infimum distance*, the number $D(X,Y)$ the *maximum distance*. Also, we call the number

$$|X| = D(X,X) \qquad (3.3)$$

the *diameter* of X. Obviously,

$$|X| = sup\{\delta(x, x') : x, x' \in X\}. \qquad (3.4)$$

We can also interpret an element X in R as a *piece of information* (equivalently as a *constraint*) about an unknown point $x \in M$. Indeed, we can imagine that the available information about x only permit to characterize a set X of possible points, i.e. to establish that x belongs to X. Consequently, if X and Y represent the available information about the points x and y, the interval $[d(X,Y),D(X,Y)]$ represents the available information about the unknown distance $\delta(x,y)$. In other words, we interpret the numbers $d(x,y)$ and $D(x,y)$ as lower-approximation and upper-approximation of the actual distance. Accordingly, we can interpret the diameter $|X|$ as a measure of the degree of incompleteness of X. Indeed, if $|X| = 0$, then the piece of information X is complete, i.e. X is a point. In the following proposition we list some basic properties of the proposed functions.

Proposition 3.1. *Let* (M,δ) *be a pseudo-metric space. Then, for any X, Y, Z bounded and nonempty subsets of M,*

(1) $d(X,X) = 0$,
(2) $d(X,Y) = d(Y,X)$ *and* $D(X,Y) = D(Y,X)$,
(3) $0 \le D(X,Y) - d(X,Y) \le |X|+|Y|$,
(4) $D(X,Y) \le D(X,Z) + D(Z,Y)$ *and* $d(X,Y) \le d(X, Z) + d(Z,Y) + |Z|$,
(5) $X \subseteq X'$, $Y \subseteq Y' \Rightarrow d(X,Y) \ge d(X',Y')$, $D(X,Y) \le D(X',Y')$,

In the case (M,δ) *is a metric space,*

(6) $D(X,Y) = 0 \Rightarrow X = Y$.

Proof. (1), (2), (5) and (6) are immediate. To prove (3), let $x, x' \in X$ and $y, y' \in Y$, then

$$\delta(x,y) \le \delta(x,x') + \delta(x',y') + \delta(y', y) \le |X| + \delta(x',y') + |Y|.$$

Consequently,

$$\delta(x,y) \le inf\{\delta(x', y') : x' \in X,\ y' \in Y\} + |X| + |Y| = d(X,Y) + |X| + |Y|$$

and therefore

$$D(X,Y) = sup\{\delta(x, y) : x \in X,\ y \in Y\} \le d(X,Y) + |X| + |Y|.$$

Since it is evident that $d(X,Y) \le D(X,Y)$, we can conclude that

$$0 \le D(X,Y) - d(X,Y) \le |X| + |Y|.$$

To prove the first inequality in (4), observe that, for any $x \in X$, $y \in Y$, $z \in Z$,

$$\delta(x, y) \le \delta(x,z) + \delta(z,y) \le D(X, Z) + D(Z, Y)$$

and so

$$D(X, Y) \le D(X, Z) + D(Z, Y).$$

Assume that $x \in X$, $y \in Y$, z and $z' \in Z$, then

$$d(X,Y) \le \delta(x,y) \le \delta(x, z) + \delta(z, z') + \delta(z', y) \le \delta(x, z) + |Z| + \delta(z', y)$$

and therefore,

$$d(X,Y) \le d(X,Z) + d(Z,Y) + |Z|. \qquad \dashv$$

4. Approximate pseudo-metric spaces

We can assume the properties listed in Proposition 3.1 as a system of axioms for point-free geometry based on "approximate distances". The resulting theory can be regarded as an extension of the theory of pseudo-metric spaces to the case of incomplete information (see also [3], [4] and [6]). So, we propose the following definition.

Definition 4.1. Let $(R, \leq)$ be a partially ordered set, $d : R \times R \to \mathbb{R}^+$ and $D : R \times R \to \mathbb{R}^+$ two functions. Then we say that $(R, \leq, d, D)$ is an *approximate pseudo-metric space*, in brief *apm-space* if the following axioms are satisfied:

A1) $d(x,x) = 0$

A2) $d(x,y) = d(y,x)$ and $D(x,y) = D(y,x)$

A3) $0 \leq D(x,y) - d(x,y) \leq |x|+|y|$

A4) $d(x,y) \leq d(x,z)+d(z,y)+|z|$ and $D(x,y) \leq D(x,z)+D(z,y)$

A5) $x \leq x',\ y \leq y' \ \Rightarrow\ d(x,y) \geq d(x',y')$ and $D(x,y) \leq D(x',y')$

where $|x| = D(x,x)$. We say that $(R, \leq, d, D)$ is an *approximate metric* space, in brief *am-space*, if

A6) $D(x,y) = 0 \Rightarrow x = y$.

We interpret the elements in R as *regions* in a geometrical space and $\leq$ as the inclusion relation. Alternatively, we interpret the elements in R as *pieces of information* or *constraints* and $\leq$ as the order with respect to the completeness of the information. The functions d, D and $|\ |$ are called *minimum distance*, *maximum distance* and *diameter*, respectively. An *isometry* from an *apm*-space $(R, \leq, d, D)$ into an *apm*-space $(R', \leq', d', D')$ is a map $h : R \to R'$ such that

(*i*) $x \leq y \Leftrightarrow h(x) \leq' h(y)$,

(*ii*) $d(x,y) = d'(h(x),h(y))$,

(*iii*) $D(x,y) = D'(h(x),h(y))$.

The class of *apm*-spaces defines a category in which the morphisms are the isometries.

Definition 4.2. A *point-region* of an *apm*-space is a region which is an atom with zero diameter. We denote by PR the set of point-regions.

There are *apm*-spaces which are *point-free*, i.e. with no point-region. Also, there are *apm*-spaces (*pm*-spaces) in which all the regions are point-regions. These spaces coincide with the pseudo-metric spaces (metric spaces).

Proposition 4.3. *The pseudo-metric spaces* (*the metric spaces*) *coincide with the apm-spaces* (*am-spaces*) *such that all the regions are point-regions. Equivalently, the pseudo-metric spaces* (*metric spaces*) *coincide with the apm-spaces* (*am-spaces*) *such that all the regions are atoms and* $d = D$.

Proof. Given a pseudo-metric space (M,δ), we can define an *apm*-space $(M, \leq, d, D)$ where $\leq$ is equal to the identity relation and $d = D = \delta$. In such a space all the regions are point-regions. Conversely, let $(R, \leq, d,D)$ be an *apm*-space in which all the regions are point-regions. Then $\leq$ is the identity relation and, by *A3*, $d = D$. By setting $\delta = d = D$, we obtain a pseudo-metric space. In a similar way we can prove the remaining part of the proposition. ⊣

By Proposition 3.1, we obtain the following basic class of *apm*-spaces.

Proposition 4.4. *Let R be a class of bounded nonempty subsets of a pseudo-metric space* (M,δ) *and define d and D by (3.1) and (3.2), respectively. Then* $(R, \subseteq, d, D)$ *is an apm-space. If* (M,δ) *is a metric space, then* $(R, \subseteq, d, D)$ *is an am-space.*

We call the spaces thus obtained *canonical*. Several interesting canonical spaces can be considered depending on the choice of R, i.e. on the *mathematical model* of the notion of region we consider appropriate. In any case, our intuition suggests that a ball and any continuous deformation of a ball is a region. Also it excludes that points, lines surfaces and other "abstract" things in an Euclidean space are regions. This in accordance with Whitehead's analysis aiming at emphasizing that objects of such a kind are the result of *"abstraction processes"*. A proper choice for these requirements is to assume that R is the class of regular bounded closed subsets.

Definition 4.5. A subset X of a pseudo-metric space (M,δ) is called *closed and regular*, in brief *regular*, provided that $X = cl(int(X))$, where *cl* and *int* are the closure and the interior operator, respectively. We call the *apm*-space defined by the class of regular bounded and nonempty subsets of M the *canonical space of the regular subsets* of (M,δ) .

In the three dimensional Euclidean space, no point, line or surface is a regular set while any continuous deformation of a closed ball is regular. Also, the following proposition shows some interesting algebraic properties of the class of regular subsets.

Proposition 4.6. *The class of regular subsets of a metric space* (M,δ) *is a complete Boolean algebra. If there is no isolated point in* (M,δ)*, then such an algebra is atom-free.*

A further motivation to confine ourselves to the regular closed sets is related to the question of the definability of identity. Observe that in a metric space (M,δ) we have that

$$x = x' \Leftrightarrow \delta(x,y) = \delta(x',y) \text{ for any } y \in M$$

and this means that the identity relation is *definable*. Now an analogous property is not true at all for the canonical spaces. In fact, if X is an open set, then the behaviour of X and its closure $cl(X)$ with respect to the minimum and the maximum distance is the same.

Proposition 4.7. *In the canonical space of the regular subsets of a metric space identity is definable. Namely, we have that, for any pair X and X' of regions:*

$$X = X' \Leftrightarrow d(X,Y) = d(X',Y) \text{ for any region } Y.$$

Proof. Assume that $d(X,Y) = d(X',Y)$ for any region Y and, by absurd, that $X \neq X'$, for example that X is not contained in X'. Then, a point $P \in X$ exists such that $P \notin X'$ and therefore such that $\delta(P,X') \neq 0$. Set $r = \delta(P,X')/2$ and let B the open ball centered in P and radius r. Then, since $X = cl(int(X))$, $B \cap int(X)$ is an open nonempty set. Set $Y = cl(B \cap int(X))$, then Y is a regular bounded nonempty set such that $d(X,Y) = 0$ and

$$d(X',Y) = d(X', B \cap int(X)) \geq d(X',B) \geq \delta(X',P) - r = \delta(P,X')/2 \neq 0.$$

This contradicts the hypothesis $d(X,Y) = d(X',Y)$. ⊣

5. Approximate metric spaces arising from classical logic

Denote by $\mathbb{F}$ the set of *sentences* of a language in first order classical logic. We call any subset of $\mathbb{F}$ a *system of axioms* and we denote by $\vdash$ the *derivability relation*, i.e., for any $\alpha \in \mathbb{F}$ and $X \subseteq \mathbb{F}$, $X \vdash \alpha$ means that we can derive α from X. Given two systems of axioms X and Y, we define the following sets of sentences whose logical meaning is evident:

$N(X) = \{\alpha \in \mathbb{F} : X \vdash \alpha\}$ (*necessary* formulas given X)

$I(X) = \{\alpha\in\mathbb{F} : X\vdash\neg\alpha\}$ (*impossible* formulas given X)
$P(X) = -I(X)$ (*possible* formulas given X)
$Pn(X) = -N(X)$ (formulas it is *possible to negate* given X)
$Dec(X) = N(X)\cup I(X)$ (formulas *logically determined* by X)
$Undec(X) = -Dec(X) = Pn(X)\cap P(X)$ (formulas *not logically determined* by X)
$Disagree(X,Y) = (N(X)\cap I(Y))\cup(N(Y)\cap I(X))$ (formulas in which X and Y *disagree*)
$Pdisagree(X,Y) = (P(X)\cap Pn(Y))\cup(P(Y)\cap Pn(X))$ (formulas in which X and Y *can disagree*)
$Agree(X,Y) = (N(X)\cap N(Y))\cup(I(Y)\cap I(X))$ (formulas in which X and Y *agree*)
$Pagree(X,Y) = (P(X)\cap P(Y))\cup(Pn(Y)\cap Pn(X))$ (formulas in which X and Y *can agree*).

According to the monotony of classical logic, we have that *N*, *I*, *Dec*, *Disagree*, *Agree* are order-preserving and *Pn*, *Undec*, *Pdisagree*, *Pagree* are order-reversing operators. Moreover, these operators are compatible with the logical equivalence, for example,

$$X \equiv X' ,\ Y \equiv Y' \Rightarrow Disagree(X,Y) = Disagree(X',Y')$$
$$X \equiv X' ,\ Y \equiv Y' \Rightarrow Pdisagree(X,Y) = Pdisagree(X',Y').$$

The proof of the following proposition is a matter of routine.

Proposition 5.1. *Let X, X′ be consistent sets of formulas, then*

$$\{Agree(X,X'),\ Disagree(X,X'),\ Undec(X)\cup Undec(X')\}$$

is a partition of the set $\mathbb{F}$ *of formulas. Moreover,*

1. $Pdisagree(X,X') = Disagree(X,X')\cup(Undec(X)\cup Undec(X')) = \mathbb{F}\text{-}Agree(X,X')$
2. $Disagree(X,X') = Pdisagree(X,X')\cap Dec(X)\cap Dec(X')$
3. $Disagree(X,X) = \varnothing$
4. $Pdisagree(X,X) = Undec(X)$.

Finally, for any consistent set of formulas Z, the following "triangular properties" old

5. $Disagree(X,X') \subseteq Disagree(X,Z)\cup Disagree(Z,X')\cup Undec(Z)$
6. $Pdisagree(X,X')\subseteq Pdisagree(X,Z)\cup Pdisagree(Z,X')$.

Now, we will consider a sentence α with respect to its information content, i.e. with respect to the information we can obtain from an answer to a query about α. From this point of view we cannot distinguish two logically equivalent sentences or a sentence and its negation. So, we have to refer to the set

$$<\alpha> = \{\alpha' \in\mathbb{F} : \text{either } \alpha'\equiv \alpha \text{ or } \alpha' \equiv\neg\alpha\}.$$

We say that $<\alpha>$ is a *test.* For every set X of sentences, we denote by $<X>$ the corresponding set $\{<\alpha> : \alpha \in X\}$ of tests. In the case α is either a logically true or logically false formula, we say that $<\alpha>$ is the *vacuous test*.

Definition 5.2. Assume that the map $rel : <\mathbb{F}> \to [0,1]$ assumes the value 0 in the vacuous test and that

$$\textstyle\sum_{x\in<\mathbb{F}>} rel(x) = 1. \tag{5.1}$$

Then we call *rel* a *relevance measure*. We call *relevant* a test *x* such that $rel(x) \neq 0$, we say that *rel* is *sensitive* provided that all the non vacuous tests are relevant.

We interpret the number $rel(<\alpha>)$ as the *degree of relevance* of the test $<\alpha>$ or, equivalently, of the formula α. The function *rel* defines a measure $\mu : P(<\mathbb{F}>)\to[0,1]$ obtained by setting, for every subset X of $<\mathbb{F}>$,

$$\mu(X) = \sum_{x\in X} rel(x). \tag{5.2}$$

We interpret $\mu(X)$ as a measure of the degree of relevance of the set X of tests.

Definition 5.3. Given a relevance measure *rel*, we define the maps $d : P(\mathbb{F})\times P(\mathbb{F}) \to [0,1]$ and $D : P(\mathbb{F})\times P(\mathbb{F}) \to [0,1]$ by setting, for any X and X' systems of axioms,

$$d(X,X') = \mu(<Disagree(X,X')>) \tag{5.3}$$

and

$$D(X,X') = \mu(<Pdisagree(X,X')>). \tag{5.4}$$

Then $d(X,X')$ is a measure of the *actual* contrast between X and X' and $D(X,X')$ is a measure of the *possible* contrast between X and X'. In both the cases the degree of relevance of the formulas is taken in account.

Theorem 5.4. *Let CA be the class of all the consistent systems of axioms, let ≤ be the reverse of the inclusion relation and define* $d(X,Y)$ *and* $D(X,Y)$ *by (5.3) and (5.4), respectively. Then the structure* $(CA, \leq, d, D)$ *is an apm-space. If the relevance measure is sensitive, then such a structure is an am-space.*

Proof. Firstly, observe that, since $Disagree(X,X')$ is contained in $Pdisagree(X,X')$, $d(X,Y) \leq D(X,Y)$. *A1*, *A2* and *A5* are immediate. *A3* follows from the fact that

$$Pdisagree(X,X') - Disagree(X,X') = Undec(X)\cup Undec(X').$$

A4 follows from 5) and 6) of Proposition 5.1.

Assume that *rel* is sensitive and that $D(X,X') = 1- \mu(<Agree(X,X')>) = 0$. Then $Agree(X,X') = \mathbb{F}$ and X and X' are complete theories such that $X = X'$. ⊣

Since $D(X,X) = 1 - \mu(<Dec(X)>) = \mu(<Undec(X)>)$, the *diameter* of X is defined by

$$|X| = \mu(<Undec(X)>) \tag{5.5}$$

i.e. $|X|$ is a measure of the degree of incompleteness of X. It is also interesting to observe that in the space of all the consistent systems of axioms we have that:

$d(X, X') = 1 \iff X$ and X' disagree in all the relevant claims
$d(X, X') = 0 \iff$ there is no relevant claim in which X and X' disagree
$D(X, X') = 1 \iff$ there is no relevant claim in which X and X' agree
$D(X, X') = 0 \iff$ in all the relevant claims X and X' agree
$|X| = 0 \iff$ any relevant claim is logically determined by X
$|X| = 1 \iff$ no relevant claim is logically determined by X.

As an example, consider the trivial case in which there is only one relevant test $<\alpha>$. Then

$$d(X, X') = \begin{cases} 1 \text{ if } X \text{ and } X' \text{ disagree on } \alpha, \\ 0 \text{ otherwise.} \end{cases} \qquad D(X, X') = \begin{cases} 0 \text{ if } X \text{ and } X' \text{ agree over } \alpha, \\ 1 \text{ otherwise.} \end{cases}$$

and

$$|X| = \begin{cases} 0 & \text{if } \alpha \text{ is logically determined by } X \\ 1 & \text{otherwise.} \end{cases}$$

The following proposition will be useful in the sequel.

Proposition 5.5. *Given two systems of axioms X and Y,*

$$D(X,Y) = d(X,Y)+|X|+|Y|-\mu(<Undec(X)\cap Undec(Y)>). \tag{5.6}$$

In particular, if Y is complete,

$$D(X,Y) = d(X,Y)+|X|. \tag{5.7}$$

Proof. By Proposition 5.1 and since $Disagree(X,Y)\cap(Undec(X)\cup Undec(Y)) = \emptyset$, we have that

$$\begin{aligned} D(X,Y) &= \mu(<Disagree(X,Y)\cup(Undec(X)\cup Undec(Y))>) \\ &= \mu(<Disagree(X,Y)) + \mu(Undec(X)\cup Undec(Y)>) \\ &= \mu(<Disagree(X,Y)) + \mu(Undec(X)) + \mu(Undec(Y))>)-\mu(<Undec(X)\cap Undec(Y)>) \\ &= d(X,Y)+|X|+|Y|-\mu(<Undec(X)\cap Undec(Y)>). \end{aligned}$$ ⊣

6. The space of the theories

Identity is not definable in the *apm*-space of consistent systems of axioms. Indeed, the following proposition holds true.

Proposition 6.1. *We have that*

$$d(X,Y) = d(N(X),Y) = d(X,N(Y)) = d(N(X),N(Y)),$$

and

$$D(X,Y) = D(N(X),Y) = D(X,N(Y)) = D(N(X),N(Y)),$$

and therefore, for any set Y of formulas,

$$X \equiv X' \;\Rightarrow\; d(X,Y) = d(X',Y) \text{ and } D(X,Y) = D(X',Y).$$

Moreover, if the relevance measure is sensitive, then

$$X \equiv X' \Leftrightarrow d(X,Y) = d(X',Y) \text{ for any set } Y \text{ of formulas.}$$

Proof. The first part of the proposition is immediate. To prove the second part, assume that $d(X,Y) = d(X',Y)$ for any set Y of formulas and assume that $N(X)$ is not contained in $N(X')$. Then a formula α exists such that $\alpha \in N(X)-N(X')$. Since $rel(<\alpha>) \neq 0$ and $d(X,X') = d(X',X') = 0$, we have that $\neg\alpha \notin N(X')$ and therefore that $Y = X'\cup\{\neg\alpha\}$ is a consistent system of axioms. Also $d(X,Y) \geq rel(<\alpha>) > 0$. On the other hand, $d(X,Y) = d(X',Y) = 0$, an absurdity. This proves that $N(X) \subseteq N(X')$. In a similar way one proves that $N(X')\subseteq N(X)$. ⊣

Such a proposition suggests that, in order to have the definability of the identity, we have to identify two logically equivalent systems of axioms. This suggests to define a region as a complete equivalence class $[X]$ modulo the logical equivalence $\equiv$ (questa frase non si capisce: cos'è "modulo"? e qual è il verbo?). An equivalent way is to select a *representative element* in any class $[X]$ and to confine ourselves only to these representative elements. In order to do this, consider the following definition.

Definition 6.2. We call any fixed point of N a *theory*.

Then a theory is a set of formulas closed under deductions. Now, since N is a closure operator, we have that, for any set T of formulas,

$$T \text{ is a theory} \Leftrightarrow \text{ there is a set } X \text{ of axioms such that } T = N(X).$$

Now, it is evident that in any complete class of equivalence $[X]$ there is an unique theory and this theory is $N(X)$. So we can identify the set of classes of equivalence with the set of theories. The class of theories is a complete lattice with respect to the set theoretical inclusion relation. Due to the geometrical interpretation we will give, we prefer to refer to the dual of this lattice, i.e. to the reverse of the inclusion relation. This means that we set $T_1 \leq T_2$ provided that $T_1 \supseteq T_2$, i.e. provided that T_1 extends the information given by T_2.

Proposition 6.3. *The class of theories is a complete lattice with respect to the reverse* $\leq$ *of the inclusion relation. In such a lattice*
- $T_1 \vee T_2$ *is* $T_1 \cap T_2$,
- $T_1 \wedge T_2$ *is the theory generated by* $T_1 \cup T_2$,
- *the maximum is the set of logically true formulas*
- *the minimum is the inconsistent theory.*

We say that a theory T is *axiomatizable* (*finitely axiomatizable*) provided that there is a decidable (finite) system of axioms X such that $T = N(X)$. From the point of view of the effectiveness of the deduction process only the axiomatizable or finitely axiomatizable theories have some interest. In this paper we consider the class of finitely axiomatizable theories but very similar results can be given for the class of axiomatizable theories.

Proposition 6.4. *The class of the finitely axiomatizable theories is a sublattice of the lattice of the theories. Such a lattice is a Boolean algebra isomorphic to the Lindenbaum algebra of the formulas.*

The just defined classes of theories define two examples of *apm*-space of particular interest for us (o "we are particularly interested in").

Definition 6.5. Consider the subspace $(Th, \leq, d, D)$ of $(CA, \leq, d, D)$ defined by the class Th of the consistent theories. Then we say that $(Th, \leq, d, D)$ is the *apm-space of all the theories.* We denote by $(Fax, \leq, d, D)$ the subspace of $(Th, \leq, d, D)$ defined by the class Fax of the finitely axiomatizable theories. We denote $(Compl, \leq, d, D)$ by the subspace defined by that class $Compl$ of the complete theories.

Observe that in $(Compl, \leq, d, D)$ the relation $\leq$ is the identity and $d = D$. Moreover

$$d(C_1,C_2) = \mu(<(C_1-C_2)\cup(C_2-C_1)>).$$

Then, as a matter of fact, such a space is a metric space.

Coming back to the question of the definability of identity, we prove the following proposition.

Proposition 6.6. *Assume that the relevance measure rel is sensitive. Then the identity is definable in both the spaces* $(Th, \leq, d, D)$ *and* $(Fax, \leq, d, D)$.

Proof. The definability of the identity in $(Th, \leq, d, D)$ is a consequence of Proposition 6.1. To show that this is true in (Fax, ≤, d, D), we have to modify the proof of Proposition 6.1 by assuming that X' is a finitely axiomatizable theory and by observing that the theory generated by $X' \cup \{\neg\alpha\}$ is finitely axiomatizable. ⊣

We can read several notions and results about the logical theories in geometrical terms. As an example, the following proposition holds true.

Proposition 6.7. *In the space* (Th, ≤, d, D) *any region is the join of its point-regions. Moreover,*

$d(T_1,T_2) = 0 \Leftrightarrow$ *T_1 and T_2 overlap*

$\Leftrightarrow$ *there is a point-region contained in both T_1 and T_2* $\Leftrightarrow$ *$T_1 \cup T_2$ is consistent.*

Two theories T_1 and T_2 are *tangent* provided that there is only one point-region contained in both T_1 and T_2.

Proposition 6.8. Two theories T_1 and T_2 are tangent if and only if $T_1 \cup T_2$ is a consistent and complete system of axioms.

We say that a theory T is *essentially incomplete* if it is incomplete and every extension of T by a finite number of formulas is again incomplete.

Proposition 6.9. The following claims are equivalent

(i) T is essentially incomplete

(ii) no region in Fax is tangent to T

(iii) in T there are $2^{\aleph_0}$ point-regions in the space Th and no point-region in the space Fax.

There are also theories whose set of point-regions is enumerable, and theories with only a finite number of point-regions. These theories are called *$\aleph_0$-complete* and *virtually complete*, respectively (see, for example [9]).

7. Verisimilitude as an approximate distance

Consider a language for the scientific discipline under consideration and a relevance measure *rel*. Now we are able to propose a notion of verisimilitude by referring to the *apm*-space of theories in such a language. Indeed, denote by $\mathbb{V}$ the set of true sentences, then we can define the verisimilitude of a theory T as a function of the distances $d(T, \mathbb{V})$ and $D(T, \mathbb{V})$. Equivalently, since by the completeness of $\mathbb{V}$,

$$D(T, \mathbb{V}) = d(T, \mathbb{V}) + |T|,$$

we can define the verisimilitude as a function of $d(T, \mathbb{V})$ and $|T|$. To simplify the argument, assume the linearity, i.e. that

$$Vs(T) = a \cdot d(T, \mathbb{V}) + b \cdot |T| + c, \qquad (7.1)$$

where, since $Vs(T)$ has to be order-reversing with respect to both $d(T, \mathbb{V})$ and $|T|$, we have to assume that a and b are negative numbers. Again, it is reasonable to assume that $Vs(T)$ attains 1 as a maximum in the case $T = \mathbb{V}$, i.e. in the case $d(T, \mathbb{V}) = 0$ and $|T| = 0$. This entails that $c = 1$. So, we propose the following quantitative definition of verisimilitude.

Definition 7.1. A *verisimilitude measure* is a map $Vs : R \to I\!R$ defined by

$$Vs(T) = 1- (t\cdot d(T, I\!V)+i\cdot |T|), \tag{7.2}$$

where $0<i\leq t\leq 1$.

Observe that $Vs(T)\in[0,1]$. Indeed, trivially $Vs(T)\leq 1$ and, since $d(T, I\!V)+|T|\leq 1$ and therefore $t\cdot d(T, I\!V)+i\cdot|T|\leq 1$, it is also $Vs(T)\geq 0$. The numbers t and i correspond to the weight we will assign to the closeness of T to the truth and to the completeness of the information carried on by T, respectively. To justify condition $i\leq t$, consider the trivial case in which there is only a relevant test $<\alpha>$ and that $\alpha\in I\!V$. Then, for any theory T,

- $d(T, I\!V) = 1$ if $\neg\alpha\in T$ and $d(T, I\!V) = 0$ otherwise,
- $|T| = 0$ if α is logically determined by T and $|T| = 1$ otherwise.

Also,

$$Vs(T) = \begin{cases} 1\text{-}t & \text{if } \neg\alpha\in T, \\ 1 & \text{if } \alpha \in T, \\ 1\text{-}i & \text{if } \alpha \text{ is not logically determined by } T. \end{cases}$$

Assume that T_1 and T_2 are two theories such that $\neg\alpha\in T_1$ while α is not logically determined by T_2. Then it is reasonable to assume that $Vs(T_1) \leq Vs(T_2)$ and this is possible only in the case $i\leq t\leq 1$.

By setting $t = 1$ and $i = ½$, we have that

$$Vs(T) = 1- (d(T, I\!V)+|T|/2). \tag{7.3}$$

In such a case $Vs(T)$ is related with the *"average distance"*, since

$$Vs(T) = 1- (d(T, I\!V)+D(T, I\!V))/2 \tag{7.4}$$

By setting $i = t = 1$, we obtain

$$Vs(T) = 1- (d(T, I\!V)+|T|). \tag{7.5}$$

or, equivalently, being $Agree(T, I\!V)$, $Disgree(T, I\!V)$, $Undec(T)$ a partition of the set of formulas,

$$Vs(T) = \mu(<Agree(T, I\!V)>). \tag{7.6}$$

We can obtain an apparently different definition of verisimilitude by referring to the notions of falsity content and truth content of a theory (see, for example, [2]). Indeed, set

$$False(T) = \{\alpha \in T : \neg\alpha\in I\!V\} \;;\; True(T) = \{\alpha \in T : \alpha\in I\!V\},$$

and define the *falsity content* and *truth content* of T, by,

$$ct_F(T) = \mu(<False(T)>) \quad ; \quad ct_V(T) = \mu(<True(T)>) \tag{7.7}$$

Then, we can propose the following definition.

Definition 7.2. A *content-based verisimilitude measure* is a map $Vs' : R \to I\!R$ defined by

$$Vs'(T) = a\cdot ct_V(T) - b\cdot ct_F(T)+c \tag{7.8}$$

where a, b and c are positive numbers.

As an example, if we set $a = b = 1$ and $c = 0$, we obtain

$$Vs'(T) = ct_V(T) - ct_F(T). \tag{7.9}$$

Proposition 7.3. *The measures of verisimilitude in Definition 7.1 and Definition 7.2 differ only* *by a linear transformation.*

Proof. It is sufficient to observe that $ct_F(T) = d(T, \mathbb{V})$ and $ct_V(T) = 1\text{-}d(T, \mathbb{V})\text{-}|T|$. ⊣

As an example, if we define $Vs(T)$ by (7.3), and Vs' by (7.9), then

$$Vs(T) = (Vs'(T)+1)/2. \tag{7.10}$$

It is also interesting to see how verisimilitude varies in adding new information.

Proposition 7.4. *Assume that* $T' \supseteq T$, *then*

$$Vs(T') = Vs(T) \text{ -} t\cdot\mu(\{<\gamma> : \gamma \in T'\text{-}T \text{ and } \gamma \text{ is false}\}) + i\cdot\mu(<Dec(T')\text{-}Dec(T)>),$$

where

$$\mu(<Dec(T')\text{-}Dec(T)>) \geq \mu(\{<\gamma> : \gamma \in T'\text{-}T \text{ and } \gamma \text{ is false}\}).$$

Proof. It is sufficient to observe that

$$d(T', \mathbb{V}) = d(T, \mathbb{V}) + \mu(\{<\gamma> : \gamma \in T'\text{-}T \text{ and } \gamma \text{ is false}\}) \text{ and } |T'| = |T| \text{ - } \mu(<Dec(T')\text{-}Dec(T)>).$$

Moreover, if $\gamma \in T'\text{-}T$, then $\gamma \notin T$ and, by the consistence of T', it is not possible that $\neg\gamma \in T$. So $\gamma \in Dec(T')\text{-}Dec(T)$. This proves that

$$Dec(T')\text{-}Dec(T) \supseteq T'\text{-}T \supseteq \{\gamma : \gamma \in T'\text{-}T \text{ and } \gamma \text{ is false}\}. \quad ⊣$$

We conclude this section by observing that this definition of verisimilitude, quantitative in nature, induces a corresponding definition, qualitative in nature. As an example, refer to the notion of *qualitative verisimilitude relation* as a family $(\leq_V)_{V\in Th}$ of pre-orders in Th (see [17]). Then we obtain a qualitative verisimilitude relation $(\leq_V)_{V\in Th}$ by setting, for any theory V,

$$T_1 \leq_V T_2 \iff t\cdot d(T_1,V)+i\cdot|T_1| \leq t\cdot d(T_2,V)+i\cdot|T_2|. \tag{7.11}$$

In [17] several conditions a verisimilitude relation have to satisfy are suggested (questa frase, così com'è, non significa niente, mi sa che ci manca qualcosa. Nel caso tu voglia dire: "In [17] sono suggerite diverse condizioni in cui una relazione di verisimilitudine deve essere soddisfatta" allora traduci così:
In [17] several conditions in which a verisimilitude relation has to be satisfied are suggested.
Se volevi dire un'altra cosa, dimmelo e te lo traduco meglio)
. It could be interesting to check whether these conditions are satisfied by the just defined verisimilitude relation.

8. Whithead's "abstraction processes" and the progress towards the truth

In accordance with Whitehead's ideas, in any *apm*-space we can define the "points" as a result of an abstraction process.

Definition 8.1. Let $(R, \leq, d, D)$ be a *apm*-space, then an *abstraction process* is any decreasing sequence $\langle x_n\rangle_{n\in N}$ of regions such that $lim_{n\to\infty}|x_n| = 0$. We denote by $AP(R)$ the class of abstraction processes of $(R, \leq, d,D)$.

We can define a structure of pseudo-metric space in the class $AP(R)$ of abstraction processes.

Proposition 8.2. *Let* $(R, \leq, d,D)$ *be a apm-space such that* $AP(R)\neq\emptyset$. *Then, for any pair* $\langle x_n\rangle_{n\in N}$ *and* $\langle y_n\rangle_{n\in N}$ *of abstraction processes,*

$$lim_{n\to\infty}d(x_n, y_n) = lim_{n\to\infty}D(x_n, y_n). \tag{8.1}$$

Moreover, if we define δ' by setting

$$\delta'(\langle x_n\rangle_{n\in N},\langle y_n\rangle_{n\in N}) = lim_{n\to\infty}d(x_n, y_n) = lim_{n\to\infty}D(x_n, y_n), \qquad (8.2)$$

then we obtain a pseudo-metric space $(AP(R),\delta')$.

Proof. The sequences $<d(x_n,y_n)>_{n\in N}$ and $<D(x_n,y_n)>_{n\in N}$ are order-preserving and order-reversing, respectively. So, they are convergent. On the other hand, by *A3* we have that $|D(x_n,y_n) - d(x_n,y_n)| \leq |x_n|+|y_n|$ and therefore, since $lim_{n\to\infty}(|x_n|+|y_n|) = lim_{n\to\infty}|x_n| + lim_{n\to\infty}|y_n| = 0$, that $lim_{n\to\infty}d(x_n,y_n) = lim_{n\to\infty}D(x_n,y_n)$. To prove that $(AP(R),\delta')$ is a pseudo-metric space, observe that

$$\delta'(<x_n>_{n\in N},<x_n>_{n\in N}) = lim_{n\to\infty} |x_n| = 0$$

and that, trivially, $\delta'(<x_n>_{n\in N}, <y_n>_{n\in N}) = \delta'(<y_n>_{n\in N},<x_n>_{n\in N})$. It remains to prove the triangular inequality. Now, if $\langle z_n\rangle_{n\in N}$ is an abstraction process, then

$$\delta'(\langle x_n\rangle_{n\in N},\langle y_n\rangle_{n\in N}) = lim_{n\to\infty}d(x_n, y_n) \leq lim_{n\to\infty}(d(x_n ,z_n) + d(z_n, y_n) + |z_n|)$$
$$= lim_{n\to\infty}d(x_n,z_n)+lim_{n\to\infty}d(z_n,y_n) = \delta'(\langle x_n\rangle_{n\in N},\langle z_n\rangle_{n\in N})+\delta'(\langle z_n\rangle_{n\in N},\langle y_n\rangle_{n\in N}). \quad \dashv$$

Such a proposition enables us to associate any *apm*-space with a metric space.

Definition 8.3. Let $(R, \leq, d,D)$ be an *apm*-space such that $AP(R) \neq\emptyset$ and let (M, δ) be the metric space associated with pseudo-metric space $(AP(R), \delta')$. Then we say that (M, δ) is *the metric space associated with* $(R, \leq, d,D)$. The elements in M are called *points*.

Then a point is a complete class of abstraction processes

$$[\langle x_n\rangle_{n\in N}] = \{\langle y_n\rangle_{n\in N} \in AP(R) : lim_{n\to\infty}d(y_n,x_n) = 0\}$$

and the distance between two points $P = [\langle x_n\rangle_{n\in N}]$ and $Q = [\langle y_n\rangle_{n\in N}]$ is defined by setting

$$\delta(P, Q) = lim_{n\to\infty} d(x_n, y_n).$$

Observe that the just defined process associating any *apm*-space $(R, \leq, d,D)$ with the metric space (M,δ) defines a functor from the subcategory of the *apm*-spaces admitting abstraction processes into the category of metric spaces. Also, we can interpret the process associating any metric space (M,δ) with the *apm*-space of the regular subsets of M as a functor from the category of the metric spaces into the category of the *apm*-spaces.

Coming back to the spaces of theories, it is easy to prove the following proposition.

Proposition 8.4. *Assume that rel is sensitive, then the metric space associated with the apm-space of the theories (ppm-axiomatizable theories, finitely axiomatizable theories) is isometric with the metric space of complete theories.*

The claim that our scientific knowledge can grow indefinitely and make progress towards the truth is at the basis of several philosophical discussions. Perhaps, a suitable definition of convergence for sequences of regions enables us to give a precise (and rough, obviously) definition of convergence of a sequence of theories to the truth. As an example, say that a sequence $<x_n>_{n\in N}$ of regions *converges to a region x*, in brief $lim_{n\to\infty}x_n=x$, provided that $\forall\varepsilon>0\exists m\ \forall n\geq m\ D(x_n,x)\leq\varepsilon$. Let $<x_n>_{n\in N}$ be a sequence of regions and x a region, then the proof of the following equivalences is immediate:

$$lim_{n\to\infty}x_n=x \iff lim_{n\to\infty}|x_n| = 0,\ |x| = 0,\ lim_{n\to\infty}d(x_n,x) = 0 \iff lim_{n\to\infty}D(x_n,x) = 0.$$

As a consequence, we have the following proposition.

Proposition 8.5. *The following are equivalent*

(a) *the sequence* T_1*,* T_2*, ... of theories converges to the truth* $\mathbb{V}$*,*

(b) $lim_{n\to\infty}\, c_T(T_n) = 1$ *and* $lim_{\,n\to\infty}\, c_F(T_n) = 0;$

(c) $lim_{n\to\infty} Vs(T_n) = 1$.

9. Solution of some paradoxes and further difficulties

Coming back to the weather example, consider the first paradox. We can avoid it by assuming that the sentences *h*, *r*, *w* are highly relevant with respect to the remaining sentences. As an example, we can set $rel(\langle h\rangle) = rel(\langle r\rangle) = rel(\langle w\rangle) = ¼$ and we can distribute the weight ¼ among the remaining formulas. Then, in the case $T_2 = \{\neg h, r, w\}$ and $T_1 = \{\neg h, \neg r, \neg w\}$, since T_2 and T_1 are complete we have

$$Vs(T_2) = 1\text{-}\mu(\langle False(T_2)\rangle) \geq 1\text{-}\ rel(\langle r\rangle)\text{-}rel(\langle w\rangle) = ½$$
$$> ¼ = 1\text{-}(rel(\langle h\rangle)+ rel(\langle r\rangle) + rel(\langle w\rangle)) \geq 1\text{-}\ \mu(\langle False(T_1)\rangle) = Vs(T_1).$$

So, in accordance with our intuition, $Vs(T_2) > Vs(T_1)$.

The invariance with respect to the language is attained by assuming that the relevance of a sentence is semantic in nature depending on the *meaning* of the sentence. This means that if in the language based on the propositional variables *h*, *r* and *w* we assign relevance ¼ to *h*, *r*, and *w*, then in the new language based on *h*, *m*, *a* we have to assign relevance ¼ to the formulas *h,* $(h\wedge m)\vee(\neg h\wedge\neg m)$ and $(h\wedge a)\vee(\neg h\wedge\neg a)$ and not to *h*, *m*, *a*. So, the calculus of the verisimilitude remains unchanged.

To face the question of the role of completeness, consider the case T_2 as the theory generated by $h\vee r$ and T_1 the theory generated by $h\vee r\vee w$. Then, since both the theories are true and $Undec(T_2)$ is a proper subset of $Undec(T_1)$,

$$Vs(T_2) = 1\text{-}\ |T_2| > 1\text{-}\ |T_1| = Vs(T_1).$$

Unfortunately, the definition of verisimilitude we propose seems inadequate with respect to further aspects.

***Vs* is not an objective notion.** Our definition of verisimilitude is based on the function *rel* and the values of this function are interpreted as the subjective "degree of relevance" we assign to the tests. This means that this definition is not an objective concept as Popper originally intended. On the other hand, it is a fact that, in testing a theory *T*, a scientific community does not consider its behaviour with respect to the whole class of possible sentences. The attention is addressed on only a (presumably finite) set *F* of sentences that are considered relevant. Moreover a sentence can be considered more or less relevant than another one depending on the "taste" predominating in the scientific community and on the "utility" of the sentences. The relevance measure *rel* is an attempt to represent this. Observe that even Popper has acknowledged in [16] that the problem of truthlikeness can perhaps be solved

"only by a relativization to relevant problems or even by bringing in the historical problem situation".

To avoid the accusation of subjectivism we can also try to give a different interpretation of the function *rel.* As an example, imagine a game with two players *S* (a scientist) and *N* (Nature). A move of *S* is to propose a theory *T*, a move of *N* is to submit *T* a test. Initially *S* receives 1 dollar but, after a test is proposed by *N*:

- *S* pays 0 in the case *T* is in accordance with the result of the test,
- *S* pays *t* in the case *T* fights with the result of the test,
- *S* pays *i* in the case *T* is not sufficiently powerful to say something about the test.

Then, by assuming that the mixed strategy of *N* is represented by the function *rel* (that we interpret as the frequency of the tests), the value $Vs(T)$ is the expected gain of S in playing T. In this case, it is reasonable to consider *rel* as an objective concept since it represents the actual frequency of the events we have to face. Then, under this interpretation the verisimilitude $Vs(T)$ seems to be an objective concept.

***Vs* can't take the simplicity of a system of axioms into account.** A second question arises from the fact that, in several cases, the comparison between two scientific theories is a comparison between two system of axioms with respect to their "simplicity" and "efficiency". It is unquestionable that Copernicus's theory is a radical advance with respect to the earth-centered theory in spite of the fact that these theories are equivalent from a logical point of view. On the other hand, due to this equivalence, our definition of verisimilitude cannot register this progress. Also, Proposition 6.1 shows that we cannot improve Definition 7.1 by a simple substitution of the class of theories with the class of the consistent sets of axioms.

***Vs* is based on classical logic and this logic is not adequate.** Further difficulties arise from the crisp dichotomy "false" and "true" which is on the basis of our definition. This dichotomy looks very far from the habit of the scientific community where it is usual to claim that a sentence is "approximately true", "non completely false" and so on. As an example, assume that h', r' and w' mean *"very near to be hot"*, *"moderately rainy"* and *"moderately windy"*, respectively. Then, in comparing the theory $T_2 = \{h', r', w'\}$ with the theory $T_1 = \{\neg h, \neg r, \neg w\}$ (with respect to the truth $V = \{h, r, w\}$) our intuition says that T_2 is closer to the truth than T_1. This in spite of the fact that classical logic assigns the same level of falsity to the two theories (see also the observations in [14]). In other words, the comparison between these theories does not refer to the set of (classically) true and false sentences these theories are able to produce. Instead it considers the fact that one theory is able to produce theorems that are closer to the truth than the other theory. These considerations seem to lead to the framework of multi-valued logic, approximate reasoning theory and fuzzy logic. As a first step, we will show that it is possible to extend any *apm*-space whose regions are subsets of a given set S into an *apm*-space whose regions are fuzzy subsets of S. We call *fuzzy subset* of S any map $s : S \rightarrow [0,1]$ and we say that s is *nonempty* if $x \in S$ exists such that $s(x) = 1$. In the class $F(S)$ of all the fuzzy subsets of S we define the *inclusion relation* by setting $s \subseteq s'$ provided that $s(x) \leq s'(x)$ for any $x \in S$. Given a number $\lambda \in [0,1]$, we call the set $C(s,\lambda) = \{x \in S : s(x) \geq \lambda\}$ the *λ-cut* of s.

Proposition 9.1. *Let $(R, \leq, d, D)$ be an apm-space in which R is a class of subsets of a set S and $\leq$ is the inclusion relation (the dual of the inclusion relation). Then, by setting $R^* = \{s \in F(S) : C(s,x) \in R \text{ for any } x \neq 0\}$ and, for any pair s and s' of elements in R^*,*

$$d^*(s,s') = \int_0^1 d(C(s,x), C(s',x))\delta x \quad ; \quad D^*(s,s') = \int_0^1 D(C(s,x), C(s',x))\delta x, \tag{9.1}$$

we obtain an apm-space $(R^*, \leq, d^*, D^*)$ where $\leq$ is the inclusion relation (the dual of the inclusion relation) between fuzzy subsets.

Proof. It is sufficient to observe that the definite integral is an additive and order-preserving operator. ⊣

Observe that $d^*(s,s')$ and $D^*(s,s')$ are the average of the distances d and D of the cuts of s and s'. Likewise, the diameter $|s| = D^*(s,s)$ of a fuzzy subset s is the average of the diameters of the cuts of s, i.e.,

$$|s|^* = \int_0^1 |C(s,\lambda)| d\lambda . \qquad (9.2)$$

In particular, Proposition 9.1 enables us to extend any canonical *apm*-space. If we say that a fuzzy subset s is *bounded* provided that all the cuts are bounded, then the regions in such an extension are bounded fuzzy subsets. Equations (9.1) define the minimum and maximum distances, respectively. Equation (9.2) defines the diameter of a bounded nonempty fuzzy subset.

A fuzzy logic is characterized by its lattice Th^* of fuzzy theories where a *fuzzy theory* is a fuzzy subsets $\tau : \mathbb{F} \to [0,1]$ of formulas closed under the fuzzy inference rules and containing the fuzzy set of logical axioms. An interesting fuzzy logic, we call *necessity logic*, is obtained as a canonical extension of classical logic (see, for example, [7]). In this logic the (totally consistent) theories are the fuzzy subsets of formulas whose cuts are consistent theories of classical logic. Then, in accordance with Proposition 9.1, we have the following proposition.

Proposition 9.2. *Let* (Th, ≤, d, D) *be the apm-space of the (consistent) scientific theories in a given first order language and define* d^* *and* D^* *as in* (9.1). *Then* (Th^*, ≤, d^*, D^*) *is an apm-space such that* Th^* *is the lattice of the totally consistent fuzzy theories of necessity logic.*

Then in necessity logic we can define the verisimilitude $Vs(\tau)$ of a fuzzy theory τ as in classical logic, i.e. as a linear function of the minimum distance and the diameter. It is easy to see that

$$Vs(\tau) = \int_0^1 Vs(C(s,\lambda)) d\lambda \qquad (9.3)$$

So, the problem of the verisimilitude for fuzzy theories reduces itself to the problem of the verisimilitude for classical theories.

Unfortunately, in spite of its elegance and interest, this definition of verisimilitude is not adequate. Indeed, in my opinion, necessity logic is not the appropriate tool to represent the notion of "approximately true" as usually used in the scientific community. As an example, in necessity logic if the sentences $\alpha_1,...,\alpha_n$ are proved at level λ, then all the formulas we can prove from $\alpha_1,...,\alpha_n$ can be proved at level λ, too. This is in contrast with the "heap paradox" phenomenon in which from almost completely true sentences we can derive a false sentence. It is not the task of this paper to individuate the appropriate fuzzy logic to face the problem of the verisimilitude, but we are convinced that researches in this direction will give interesting results.

References

1. Blumenthal, L.M., *Theory and Applications of Distance Geometry,* Chelsea Publishing Company, N.Y.,1970.
2. Burger, I. C., J. Heidema, Comparing theories by their positive and negative contents, *British Journal for the Philosophy of Science*, 44 (1993) 605-630.
3. Gerla, G., Volpe, R., Without points geometry, *Am. Math. Mon.* 92 (1985) 707-711.
4. Gerla, G., Pointless Metric Spaces, *J. Symb. Logic*, 55 (1990) 207-219.
5. Gerla G., Distances, diameters and verisimilitude of theories, *Archive for Mathematical Logic*, 31 (1992) 407-414.
6. Gerla, G., Pointless Geometries. In: Buekenhout, F., Kantor, W. (eds), *Handbook of incidence geometry: buildings and foundations.* Amsterdam: North-Holland, (1995) 1015-1031.

7. Gerla, G., *Fuzzy Logic*: *Mathematical Tools for Approximate Reasoning*, Kluwer Academic Press, 2001.
8. Hilpinen, R., Approximate truth and truthlikeness. In: Przelecki, M., Szaniawski, K., Wójcicki, R. (eds) *Formal methods in the methodology of empirical sciences*, vol. 103, pp. 19-42. Dordrecht: Reidel (1987).
9. De Bouvère, L.K., Remarks on classification of theories by their complete extensions, *Notre Dame J. Formal Logic* 10 (1969) 1-17.
10. Miller, D., Popper's qualitative theory of verisimilitude, *British Journal for the Philosophy of Science*, 25 (1974) 166-177.
11. Miller, D., On distance from the truth as a true distance. In: Hintikka, J., Niiniluoto, I., Saarinen, E. (eds) *Essays on Mathematical and Philosophical Logic*, vol. 122,. Dordrecht: Reidel (1979) 415-435.
12. Niiniluoto, I., *Is science progressive?* vol. 177. Dordrecht: Reidel 1984.
13. Niiniluoto, I., *Truthlikeness*. vol. 185. Dordrecht: Reidel 1987.
14. Oddie, Graham, "Truthlikeness", *The Stanford Encyclopedia of Philosophy (Fall 2001 Edition)*, Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/fall2001/entries/truthlikeness/>.
15. Popper, K.R., *Conjectures and Refutations*. Routledge and Kegan, London, 1963.
16. Popper, K.R., *Objective Knowledge.* Oxford: Oxford University Press 1979.
17. Ryan M., Schobbens P., Belief Revision and Verisimilitude, *Notre Dame J. Formal Logic* 36 (1995) 15-29.
18. Tichy, P., On Popper's definitions of verisimilitude, *British Journal for the Philosophy of Science* 25 (1974) 155-188.
19. Whitehead A. N., *An inquiry concerning the Principles of Natural Knowledge,* Univ. Press, Cambridge, 1919.
20. Whitehead A. N., *The Concept of Nature,* Univ. Press, Cambridge, 1920.
21. Whitehead A. N., *Process and Reality,* McMillian, N.Y., 1929.