DIFFERENTIAL FORMS ON MODULI SPACES OF PARABOLIC BUNDLES

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Abstract. Let X be a smooth projective variety, and let $\mathcal{PB}$ be a moduli space of stable parabolic bundles on X. For any flat family E_* of parabolic bundles on X parametrized by a smooth scheme Y, and for any integer m, with $1 \leq m \leq \dim X$, we construct a closed differential form $\Omega = \Omega_{E_*}$ on Y with values in $H^m(X, \mathcal{O}_X)$. By using the vector-valued differential form Ω we then prove that the choice of a (non-zero) differential m-form σ on X, $\sigma \in H^0(X, \Omega_X^m)$, determines, in a natural way, a closed differential m-form Ω_σ on the smooth locus of $\mathcal{PB}$.

Introduction

In this paper we want to provide another example of a general phenomenon, which can be roughly stated as follows: "geometric structures" on a base variety X determine similar structures on various kinds of moduli spaces of sheaves on X.

So, let X be a smooth projective variety, defined over an algebraically closed field k of characteristic 0. In [B2], we proved that, if X admits non-zero differential forms of degree m, then the choice of any such m-form σ determines a differential m-form Ω_σ on the smooth locus of the moduli space $\mathcal{M}$ of stable sheaves on X. Moreover, the restriction of Ω_σ to the smooth locus of the open subscheme of $\mathcal{M}$ parametrizing stable locally free sheaves is closed.

In this paper we shall prove an analogous result for moduli spaces of parabolic bundles. More precisely, let us denote by $\mathcal{PB}$ a moduli space of stable parabolic bundles on X. Then we show that the choice of a non-zero differential m-form on X determines a closed differential m-form Ω_σ on the smooth locus of the moduli space $\mathcal{PB}$.

This paper is organized as follows: in Section 1 we recall the definitions of parabolic sheaves and parabolic bundles on a higher dimensional variety X, then in Section 2 we recall some useful results about cup-products and trace maps. We also introduce the symmetrized trace map and study its graded commutativity properties. This is the main technical tool needed to construct closed differential forms on moduli spaces of stable parabolic bundles on X.

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In Section 3 we construct, for any flat family E_* of parabolic bundles on X parametrized by a smooth scheme Y, and for any integer m, with $1 \leq m \leq \dim X$, a closed differential form $\Omega = \Omega_{E_*}$ on Y with values in $H^m(X, \mathcal{O}_X)$.

Finally, in Section 4, we use the vector-valued differential form Ω to construct ordinary (i.e., scalar-valued) differential forms on the smooth locus $\mathcal{PB}^{sm}$ of the moduli space $\mathcal{PB}$ of stable parabolic bundles on X. More precisely, we prove that the choice of a (non-zero) differential m-form σ on X, $\sigma \in H^0(X, \Omega_X^m)$, determines, in a natural way, a differential m-form Ω_σ on $\mathcal{PB}^{sm}$, defined by using the vector-valued m-form Ω. Then the closure of Ω immediately implies the closure of the m-form Ω_σ.

As a special case, we can take $m = \dim X$. Then, if X has an effective canonical divisor K_X and if we take $D \in |K_X|$, there is a canonical choice of the section $\sigma \in H^0(X, K_X)$, namely the section defining the effective divisor D. It follows that, in this particular case, there is a canonical closed differential m-form on the smooth locus of the moduli space of stable parabolic bundles on X with parabolic structure over D.

1. Parabolic sheaves

In this section we shall briefly recall the definitions of parabolic sheaves and parabolic bundles on higher dimensional varieties. For more details we refer the reader to [MY] and [Bh].

Let X be a non-singular projective variety of dimension n, defined over an algebraically closed field k of characteristic 0, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X. We shall fix an effective Cartier divisor D on X.

Definition 1.1. A parabolic structure over D on a coherent, torsion-free $\mathcal{O}_X$-module E is the data of a filtration

$$F_* : \quad E = F_1(E) \supset F_2(E) \supset \cdots \supset F_l(E) \supset F_{l+1}(E) = E(-D),$$

where $E(-D)$ denotes the image of $E \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D) \to E$, together with a sequence of real numbers $\alpha_* = (\alpha_1, \ldots, \alpha_l)$, called weights, such that

$$0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_l < 1.$$

A parabolic sheaf on X is a coherent, torsion-free $\mathcal{O}_X$-module E with a parabolic structure over D.

Remark 1.2. There is another definition of parabolic sheaves, which is closer to the original definition of parabolic bundles on curves (cf., [Bh]): a parabolic structure over D on a torsion-free sheaf E is given by a sequence of subsheaves of $E|_D$

$$E|_D = \mathcal{F}_D^1(E) \supset \mathcal{F}_D^2(E) \supset \cdots \supset \mathcal{F}_D^l(E) \supset \mathcal{F}_D^{l+1}(E) = 0,$$

together with a system of weights $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_l < 1$.

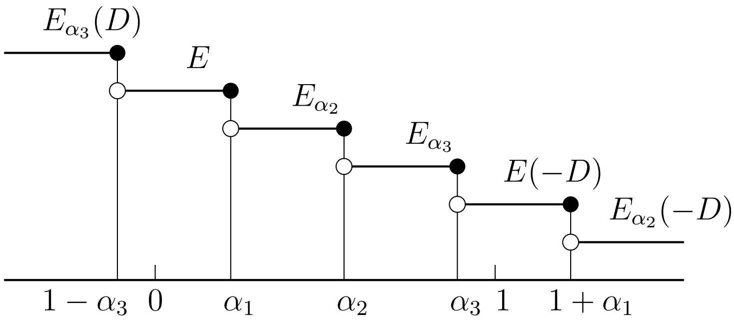

FIGURE 1. The $\mathbb{R}$-filtered sheaf E_* associated to a parabolic sheaf (E, F_*, α_*).

Our definition is related to this one by setting

$$F_i(E) = \ker(E \to E|_D/\mathcal{F}^i_D(E)).$$

All definitions related to parabolic sheaves can be stated more efficiently in terms of $\mathbb{R}$-filtered sheaves (see [Y2] for the definition), whose introduction seems to be due originally to C. Simpson.

Given a parabolic sheaf (E, F_*, α_*), we define its associated $\mathbb{R}$-filtered sheaf $E_* = (E_x)$, for $0 \le x \le 1$, by setting $E_0 = E$ and $E_x = F_i(E)$ if $\alpha_{i-1} < x \le \alpha_i$, where we have set $\alpha_0 = 0$ and $\alpha_{l+1} = 1$. The definition of E_x can be extended to all $x \in \mathbb{R}$ by setting $E_{x+1} = E_x(-D)$. Figure 1 illustrates the $\mathbb{R}$-filtered sheaf corresponding to a parabolic sheaf (E, F_*, α_*) with weights $0 \le \alpha_1 < \alpha_2 < \alpha_3 < 1$.

From now on an $\mathbb{R}$-filtered sheaf $E_* = (E_x)_{x\in\mathbb{R}}$ associated to a parabolic sheaf (E, F_*, α_*) as above, will be simply called a parabolic sheaf.

If E_* is an $\mathbb{R}$-filtered sheaf, we shall always write E for the sheaf E_0.

Definition 1.3. A homomorphism of $\mathbb{R}$-filtered sheaves $\phi : E_* \to E'_*$ is a homomorphism of $\mathcal{O}_X$-modules $\phi : E \to E'$ such that $\phi(E_x) \subseteq E'_x$, for any $x \in \mathbb{R}$.

We shall denote by $\mathcal{H}om(E_*, E'_*)$ the sheaf of homomorphisms of $\mathbb{R}$-filtered sheaves from E_* to E'_*; it is a subsheaf of $\mathcal{H}om(E, E')$. We shall also write $\mathcal{E}nd(E_*)$ for $\mathcal{H}om(E_*, E_*)$.

With these definitions the notion of parabolic homomorphism of two parabolic sheaves becomes very simple:

Definition 1.4. If E_* and E'_* are two parabolic sheaves, a parabolic homomorphism $\phi : E_* \to E'_*$ is a homomorphism of $\mathbb{R}$-filtered sheaves.

Let now Y be a locally noetherian scheme defined over k and let us denote by $p : X \times Y \to X$ and $q : X \times Y \to Y$ the canonical projections. We shall need the notion of a flat family of parabolic sheaves on X, parametrized by Y, hence we start by recalling the usual definition of a flat family of coherent sheaves:

Definition 1.5. A Y-flat family of coherent, torsion-free sheaves on X is a coherent sheaf on $X \times Y$, flat over Y such that, for any closed point y of Y, the sheaf $E_y = E|_{X\times\{y\}} = E \otimes_{\mathcal{O}_Y} k(y)$ is a torsion-free $\mathcal{O}_X$-module.

Now let us consider a Y-flat family E of coherent, torsion-free sheaves on X, and let F_* be a filtration of E and α_* a system of weights, as in Definition 1.1.

Definition 1.6. A Y-flat family of parabolic sheaves on X is a triple (E, F_*, α_*) as above, such that all the sheaves $E/F_i(E)$ are flat over Y.

If (E, F_*, α_*) is a Y-flat family of parabolic sheaves, it follows from the definition that all the subsheaves $F_i(E)$ of E are flat over Y. This implies that we can associate to (E, F_*, α_*) an $\mathbb{R}$-filtered sheaf E_* as before, hence we can denote a Y-flat family of parabolic sheaves (E, F_*, α_*) simply by its associated $\mathbb{R}$-filtered sheaf E_*.

In the sequel we shall be particularly interested in a special class of parabolic sheaves, namely locally free parabolic sheaves (also called parabolic bundles).

Definition 1.7. A parabolic sheaf E_* is said to be locally free, or a parabolic bundle, if, for any x, E_x is a locally free $\mathcal{O}_X$-module and, for any x, y, with $x \leq y < x+1$, E_x/E_y is a locally free $\mathcal{O}_D$-module.

The obvious definition of a Y-flat family of locally free parabolic sheaves (or parabolic bundles) on X is left to the reader.

2. Preliminaries on trace maps

2.1. **Cup-product and trace maps.** In this section we shall generalize, to the case of parabolic bundles, the description of trace maps given in [B2] for ordinary sheaves.

Let E_* be a parabolic bundle on X and let $\mathcal{E}nd(E_*)$ be the sheaf of parabolic endomorphisms of E_*. Since $\mathcal{E}nd(E_*)$ is a subsheaf of $\mathcal{E}nd(E)$, the usual trace map

$$\operatorname{tr} : \mathcal{E}nd(E) \to \mathcal{O}_X$$

restricts to a trace map, denoted by the same symbol,

$$\operatorname{tr} : \mathcal{E}nd(E_*) \to \mathcal{O}_X.$$

This map, in turn, induces natural maps (also called trace maps and denoted again by the same symbol)

$$\operatorname{tr} : H^i(X, \mathcal{E}nd(E_*)) \to H^i(X, \mathcal{O}_X).$$

For any i and j there is a natural cup-product (or Yoneda composition) map

$$H^i(X, \mathcal{E}nd(E_*)) \times H^j(X, \mathcal{E}nd(E_*)) \xrightarrow{\circ} H^{i+j}(X, \mathcal{E}nd(E_*))$$

and the composition of cup-product and trace is graded commutative in the following sense: if $\alpha \in H^i(X, \mathcal{E}nd(E_*))$ and $\beta \in H^j(X, \mathcal{E}nd(E_*))$, then

$$\operatorname{tr}(\alpha \circ \beta) = (-1)^{ij} \operatorname{tr}(\beta \circ \alpha) \tag{2.1}$$

as an element of $H^{i+j}(X,\mathcal{O}_X)$.

Analogous maps can also be defined in a relative situation.

Let Y be a locally noetherian scheme defined over k and denote by $p : X\times Y \to X$ and $q : X\times Y \to Y$ the canonical projections. Let E_* be a Y-flat family of parabolic bundles on X.

Then, for any i and j, there is a cup-product map

$$R^i q_* \mathcal{E}nd(E_*) \times R^j q_* \mathcal{E}nd(E_*) \xrightarrow{\circ} R^{i+j} q_* \mathcal{E}nd(E_*)$$

and a trace map

$$\mathrm{tr} : R^i q_* \mathcal{E}nd(E_*) \to R^i q_* \mathcal{O}_{X\times Y} \cong H^i(X,\mathcal{O}_X)\otimes_k \mathcal{O}_Y$$

satisfying (2.1) for any sections α and β of $R^i q_* \mathcal{E}nd(E_*)$ and $R^j q_* \mathcal{E}nd(E_*)$, respectively.

2.2. **The symmetrized trace map.** Let E_* be a parabolic bundle on X. For any integer $m \geq 1$ let us consider the "symmetrized composition map"

$$\underbrace{\mathcal{E}nd(E_*)\times\cdots\times\mathcal{E}nd(E_*)}_{m} \xrightarrow{S} \mathcal{E}nd(E_*) \tag{2.2}$$

defined by setting

$$S(\phi_1,\phi_2,\ldots,\phi_m) = \frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m} \phi_{\sigma(1)}\circ\phi_{\sigma(2)}\circ\cdots\circ\phi_{\sigma(m)},$$

where the sum runs over the group $\mathfrak{S}_m$ of permutations of m elements (for $m = 1$ we simply get the identity $S = \mathrm{id} : \mathcal{E}nd(E_*) \to \mathcal{E}nd(E_*)$).

We define the "symmetrized trace", denoted by Str, to be the composition of S with the usual trace map:

$$\mathrm{Str}(\phi_1,\phi_2,\ldots,\phi_m) = \frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m} \mathrm{tr}(\phi_{\sigma(1)}\circ\phi_{\sigma(2)}\circ\cdots\circ\phi_{\sigma(m)}). \tag{2.3}$$

The map

$$\mathrm{Str} : \mathcal{E}nd(E_*)\times\cdots\times\mathcal{E}nd(E_*) \to \mathcal{O}_X \tag{2.4}$$

is totally symmetric and multilinear.

For $m = 1$, we obviously find the usual trace map $\mathrm{tr} : \mathcal{E}nd(E_*) \to \mathcal{O}_X$. For $m = 2$, we also have $\mathrm{Str}(\phi_1,\phi_2) = \mathrm{tr}(\phi_1\circ\phi_2)$, because the usual trace satisfies the following symmetry property:

$$\mathrm{tr}(\phi_1\circ\phi_2) = \mathrm{tr}(\phi_2\circ\phi_1).$$

Again, by using this symmetry property of the trace, we see that, for $m = 3$,

$$\mathrm{Str}(\phi_1,\phi_2,\phi_3) = \frac{1}{2}\Big(\mathrm{tr}(\phi_1\circ\phi_2\circ\phi_3) + \mathrm{tr}(\phi_1\circ\phi_3\circ\phi_2)\Big).$$

In a similar way, it is easy to prove that, for a general $m \geq 2$,

$$\mathrm{Str}(\phi_1, \phi_2, \dots, \phi_m) = \frac{1}{(m-1)!} \sum_{\sigma} \mathrm{tr}(\phi_1 \circ \phi_{\sigma(2)} \circ \cdots \circ \phi_{\sigma(m)}), \tag{2.5}$$

where the sum runs over all permutations σ of the set $\{2, 3, \dots, m\}$.

The symmetrized trace map (2.4) induces a map, also denoted by Str,

$$\mathrm{Str} : H^{i_1}(X, \mathcal{E}nd(E_*)) \times \cdots \times H^{i_m}(X, \mathcal{E}nd(E_*)) \to H^{i_1+\cdots+i_m}(X, \mathcal{O}_X). \tag{2.6}$$

This map satisfies a kind of graded commutativity property similar to the one stated in (2.1).

Proposition 2.1. *Let $\phi_h \in H^{i_h}(X, \mathcal{E}nd(E_*))$, for $h = 1, \dots, m$. For any integer p, with $1 \leq p \leq m-1$, we have:*

$$\mathrm{Str}(\phi_1, \dots, \phi_p, \phi_{p+1}, \dots, \phi_m) = (-1)^{i_p i_{p+1}} \mathrm{Str}(\phi_1, \dots, \phi_{p+1}, \phi_p, \dots, \phi_m),$$

i.e., whenever we mutually exchange two adjacent *elements ϕ_p and ϕ_{p+1}, the value of* Str *acquires the factor $(-1)^{\deg(\phi_p)\deg(\phi_{p+1})} = (-1)^{i_p i_{p+1}}$.*

Proof. See [B2]. □

In the sequel we shall be interested in a special case of (2.6). By taking all i_h equal to 1, we get the map

$$\mathrm{Str} : \underbrace{H^1(X, \mathcal{E}nd(E_*)) \times \cdots \times H^1(X, \mathcal{E}nd(E_*))}_{m} \to H^m(X, \mathcal{O}_X),$$

satisfying

$$\mathrm{Str}(\phi_1, \dots, \phi_p, \phi_{p+1}, \dots, \phi_m) = -\mathrm{Str}(\phi_1, \dots, \phi_{p+1}, \phi_p, \dots, \phi_m),$$

for every $p \in [1, m-1]$.

Since every permutation σ of $\{1, 2, \dots, m\}$ may be expressed as the product of a certain number s of transpositions of adjacent elements $\pi_{p,p+1}$, and since the sign of σ is given by $\mathrm{sgn}(\sigma) = (-1)^s$, we deduce the following result:

Corollary 2.2. *For any $m \geq 1$, the map*

$$\mathrm{Str} : \underbrace{H^1(X, \mathcal{E}nd(E_*)) \times \cdots \times H^1(X, \mathcal{E}nd(E_*))}_{m} \to H^m(X, \mathcal{O}_X)$$

is alternating, i.e., for any permutation σ of $\{1, 2, \dots, m\}$, we have:

$$\mathrm{Str}(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \dots, \alpha_{\sigma(m)}) = \mathrm{sgn}(\sigma)\,\mathrm{Str}(\alpha_1, \alpha_2, \dots, \alpha_m).$$

Let now Y be a locally noetherian scheme defined over k and let E_* be a Y-flat family of parabolic bundles on X. The preceding constructions can be generalized to this relative situation (exactly as in the case of the usual trace map, described in Section 2.1). We leave the details to the reader and just state the relative version of Corollary 2.2:

Corollary 2.3. *Let E_* be a Y-flat family of parabolic bundles on X. For any $m \geq 1$, the map*

(2.7)

$$\mathrm{Str} : \underbrace{R^1 q_* \mathcal{E}nd(E_*) \times \cdots \times R^1 q_* \mathcal{E}nd(E_*)}_{m} \to R^m q_*(\mathcal{O}_{X \times Y}) \cong H^m(X, \mathcal{O}_X) \otimes_k \mathcal{O}_Y$$

is alternating, i.e., for any permutation σ of $\{1, 2, \dots, m\}$, we have:

$$\mathrm{Str}(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \dots, \alpha_{\sigma(m)}) = \mathrm{sgn}(\sigma)\,\mathrm{Str}(\alpha_1, \alpha_2, \dots, \alpha_m).$$

Remark 2.4. In this section we have assumed, for simplicity of exposition, that E_* is a parabolic bundle (or a family of parabolic bundles) on X. If, more generally, E_* is a parabolic sheaf on X (i.e., if it is not necessarily locally free), it is necessary to replace the cohomology groups $H^i(X, \mathcal{E}nd(E_*))$ with the "parabolic Ext" groups $\mathrm{Ext}^i(E_*, E_*)$ (see [Y2] for the definition). The construction of trace maps, cup-products and symmetrized trace maps can be naturally extended to this more general setting (cf. [B2] for more details).

3. Vector-valued differential forms

Let Y be a *smooth* scheme of finite type over k and E_* a Y-flat family of parabolic bundles on X. In this section we shall define, for any such E_* and any integer m, with $1 \leq m \leq n = \dim X$, a vector-valued m-form on Y (more precisely, a differential form of degree m on Y with values in $H^m(X, \mathcal{O}_X)$). Then we shall prove that these differential forms are closed.

Let us begin by recalling that, for any parabolic bundle F_* on X, the set of isomorphism classes of infinitesimal deformations of F_* is canonically identified with $H^1(X, \mathcal{E}nd(F_*))$. It follows that, for any family E_* of parabolic bundles on X parametrized by Y, there is a map, called the Kodaira-Spencer map,

$$\rho : TY \to R^1 q_* \mathcal{E}nd(E_*), \tag{3.1}$$

that sends a tangent vector $v \in T_y Y$ to the class $\rho(v) \in H^1(X, \mathcal{E}nd(E_{*y}))$ corresponding to the infinitesimal deformation of the parabolic bundle E_{*y} along the direction of v.

Now, for any m as above, we define an $H^m(X, \mathcal{O}_X)$-valued differential m-form $\Omega = \Omega_{E_*}$ on Y by setting

$$\Omega : \underbrace{TY \times \cdots \times TY}_{m} \to R^1 q_* \mathcal{E}nd(E_*) \times \cdots \times R^1 q_* \mathcal{E}nd(E_*) \to H^m(X, \mathcal{O}_X) \otimes_k \mathcal{O}_Y,$$

where the first map is induced by the Kodaira-Spencer map (3.1), and the second one is the symmetrized trace map (2.7). In other words, we set

$$\Omega(v_1, \dots, v_m) = \mathrm{Str}(\rho(v_1), \dots, \rho(v_m)), \tag{3.2}$$

for any sections $v_1, \dots, v_m$ of the tangent bundle TY. It follows from Corollary 2.3 that Ω is a vector-valued differential form of degree m.

Remark 3.1. Let E_* be a Y-flat family of parabolic bundles on X and L be a line bundle on Y. We can define another Y-flat family of parabolic bundles E'_* on X by setting $E'_* = E_* \otimes q^*(L)$. These two families of parabolic bundles may be considered as equivalent because, for every closed point $y \in Y$, the parabolic bundles E_{*y} and E'_{*y} on X are isomorphic. Under these hypotheses, the differential m-forms Ω_{E_*} and $\Omega_{E'_*}$ are equal.

Remark 3.2. Let us observe that in the definition of Ω_{E_*} we do not use directly the parabolic bundle E_*, but rather the sheaf $R^1 q_* \mathcal{E}nd(E_*)$. This is very important because in most interesting applications, when we take as Y a suitable moduli space of stable parabolic bundles on X, a global universal family of parabolic bundles E_* does not exist, but the sheaf $R^1 q_* \mathcal{E}nd(E_*)$ on Y is, nevertheless, well defined (cf. Remark 4.3). It follows that our definition of the differential form Ω_{E_*} remains valid also in these more general situations.

Now the proof of the closure of the vector-valued differential form Ω_{E_*} is essentially the same as the one given in [B2].

4. Differential Forms on Moduli Spaces

In this section we shall apply the results of Section 3 to the construction of closed differential forms on moduli spaces of parabolic bundles.

Let X be a non-singular projective variety defined over an algebraically closed field k of characteristic 0 and $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X.

In order to construct moduli spaces of parabolic sheaves we need, as usual, a suitable notion of stability. This was introduced in [MY], where moduli spaces of semistable parabolic sheaves were constructed in great generality. We only state here the results we shall need in the sequel.

Proposition 4.1. *Let us fix a sequence of rational numbers $\alpha_* = (\alpha_1, \dots, \alpha_l)$ with $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_l < 1$, and polynomials $H, H_1, \dots, H_l$. Then there exists a quasi-projective moduli space $\mathcal{PS}$ parametrizing isomorphism classes of stable parabolic sheaves E_* having α_* as system of weights and such that the Hilbert polynomial of E is H and the Hilbert polynomial of $E/F_{i+1}(E)$ is H_i, for $i = 1, \dots, l$.*

Let us restrict our attention to the open subset $\mathcal{PB}$ of the moduli space $\mathcal{PS}$ which parametrizes isomorphism classes of parabolic bundles.

Infinitesimal deformation theory for parabolic sheaves (cf. [Y2]) yields the following result:

Proposition 4.2. *The tangent space $T_{E_*}\mathcal{PB}$ to the moduli space $\mathcal{PB}$ at a point E_* is canonically identified with the cohomology group $H^1(X, \mathcal{E}nd(E_*))$ and the obstruction to the smoothness of $\mathcal{PB}$ at the point E_* lies in $H^2(X, \mathcal{E}nd(E_*))$.*

The moduli space $\mathcal{PB}$ is, in general, not smooth. We shall denote by $\mathcal{PB}^{sm}$ its smooth locus.

Remark 4.3. On the moduli space $\mathcal{PB}$ there does not exist, in general, a universal family of parabolic bundles $\mathcal{E}_*$, not even locally in the Zariski topology. In any case, a universal family $\mathcal{E}_*$ on $\mathcal{PB}$ exists locally in the étale topology (or in the complex analytic topology, if $k = \mathbb{C}$). As noted in Remark 3.1, these local universal families are not uniquely determined, in fact they are defined only up to tensoring with the pull-back of a line bundle on $\mathcal{PB}$. In general, these ambiguities prevent the local universal families to glue together to a globally defined one. On the other hand, when we consider the sheaf of parabolic endomorphisms $\mathcal{E}nd(\mathcal{E}_*)$, these ambiguities disappear, and these locally defined sheaves glue together to a globally defined one on $\mathcal{PB}$. For this reason, we shall abuse the notation and write $\mathcal{E}nd(\mathcal{E}_*)$ even if the universal family $\mathcal{E}_*$ does not exist on $\mathcal{PB}$.

With these notations, we can state the global version of Proposition 4.2:

Proposition 4.4. *Let $\mathcal{E}_*$ be a (locally defined) universal family of parabolic bundles on $\mathcal{PB}^{sm}$. Then we have a natural isomorphism (given by the Kodaira-Spencer map)*

$$T\mathcal{PB}^{sm} \cong R^1 q_* \, \mathcal{E}nd(\mathcal{E}_*).$$

We can now apply the results of the preceding section to construct natural differential forms on the moduli space $\mathcal{PB}^{sm}$.

More precisely, by setting $Y = \mathcal{PB}^{sm}$ and denoting by $\mathcal{E}_*$ a locally defined universal family on Y, we have, for any m with $1 \le m \le n = \dim X$, a vector-valued m-form

$$\Omega : T\mathcal{PB}^{sm} \times \cdots \times T\mathcal{PB}^{sm} \to H^m(X, \mathcal{O}_X) \otimes_k \mathcal{O}_{\mathcal{PB}^{sm}}. \tag{4.1}$$

Let us now assume that there exists a differential m-form σ on X, $\sigma \in H^0(X, \Omega^m_X)$. The multiplication by σ defines a map

$$H^m(X, \mathcal{O}_X) \xrightarrow{\sigma} H^m(X, \Omega^m_X). \tag{4.2}$$

Finally, if we denote by $\eta_X \in H^1(X, \Omega^1_X)$ the cohomology class of the polarization $\mathcal{O}_X(1)$, we have a map

$$H^m(X, \Omega^m_X) \xrightarrow{\eta_X^{n-m}} H^n(X, \Omega^n_X) \cong k. \tag{4.3}$$

By composing the vector-valued differential form Ω with the maps (4.2) and (4.3), we obtain an ordinary (scalar-valued) m-form, which we denote by Ω_σ:

$$\Omega_\sigma : T\mathcal{PB}^{sm} \times \cdots \times T\mathcal{PB}^{sm} \to \mathcal{O}_{\mathcal{PB}^{sm}}.$$

Since the vector-valued form Ω is closed, it follows that Ω_σ is a closed differential m-form.

We can summarize these results as follows:

Theorem 4.5. *For any differential m-form σ on X there is a closed differential m-form Ω_σ on the smooth locus $\mathcal{PB}^{sm}$ of the moduli space of stable parabolic bundles on X.*

Remark 4.6. The construction of the differential m-form Ω (hence also of Ω_σ) can be extended to the smooth locus $\mathcal{PS}^{sm}$ of the moduli space $\mathcal{PS}$ of stable parabolic sheaves. Then, whenever $\mathcal{PB}^{sm}$ is a dense open subscheme of $\mathcal{PS}^{sm}$, the closure of Ω on $\mathcal{PB}^{sm}$ obviously implies the closure of Ω on $\mathcal{PS}^{sm}$. We conjecture that this is always true, i.e., that the m-form Ω on $\mathcal{PS}^{sm}$ is always closed.

To end this section, let us describe some particularly interesting special cases of Theorem 4.5.

Let us take $m = n = \dim X$ and assume that there exists a non-zero section σ of the canonical line bundle $K_X = \Omega^n_X$. In this case the map (4.3) is the identity, hence the n-form Ω_σ is given by the composition of Ω in (4.1) with the map

$$H^n(X, \mathcal{O}_X) \xrightarrow{\sigma} H^n(X, K_X) \cong k.$$

A particularly interesting situation arises when X has an effective canonical divisor and we take $D \in |K_X|$. In this case there is a canonical choice (up to a scalar multiple) of a section $\sigma \in H^0(X, K_X)$, namely the section defining the effective divisor D. It follows that, on the moduli space $\mathcal{PB}^{sm}$ of stable parabolic bundles on X with parabolic structure over D there is a canonical closed differential n-form Ω_σ.

Another special case where there is a canonical choice of the section σ is when the canonical line bundle of X is trivial. In this case the natural choice for σ is $\sigma = 1 \in H^0(X, K_X) \cong k$. It follows that, in this case too, there is a canonical n-form on the moduli space $\mathcal{PB}^{sm}$.

References

[Bh] U. Bhosle, *Parabolic sheaves on higher dimensional varieties*, Math. Ann. **293** (1992), 177–192.

[B1] F. Bottacin, *Poisson structures on moduli spaces of parabolic bundles on surfaces*, Manuscripta Math. **103** (2000), 31–46.

[B2] ———, *Closed differential forms on moduli spaces of sheaves*, preprint (2006).

[MY] M. Maruyama, K. Yokogawa, *Moduli of parabolic stable sheaves*, Math. Ann. **293** (1992), 77–99.

[Y1] K. Yokogawa, *Compactification of moduli of parabolic sheaves and moduli of parabolic Higgs sheaves*, J. Math. Kyoto Univ. **33** (1993), 451–504.

[Y2] ———, *Infinitesimal deformations of parabolic Higgs sheaves*, Internat. J. Math. **6** (1995), 125–148.

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