ATIYAH CLASSES AND CLOSED FORMS ON MODULI SPACES OF SHEAVES

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Abstract. Let X be a smooth projective variety, and let Y be a moduli space of stable sheaves on X. By using the local Atiyah class of a universal family of sheaves on Y, which is well defined even when such a universal family does not exist, we are able to construct a natural map associating to any differential form of degree k on X or, more generally, to any cohomology class $\sigma \in H^i(X, \Omega_X^{i+k})$, for $i \geq 0$, a closed differential form of degree k on Y. This method provides a natural way to construct closed differential forms on moduli spaces of sheaves. We remark that no smoothness hypothesis is made on the moduli space Y.

Introduction

Let X be a smooth projective variety, defined over an algebraically closed field k of characteristic 0, and let Y be a moduli space of stable sheaves on X. Usually Y will be a singular quasi-projective variety, whose points parametrize isomorphism classes of stable coherent sheaves of $\mathcal{O}_X$-modules with a fixed Hilbert polynomial.

It is by now well known that many geometric properties of the moduli space Y are determined by similar properties of the variety X. One of the first examples was discovered by S. Mukai in [Mu1], where he proved that if X is a symplectic algebraic surface (i.e., an abelian or K3 surface), one can construct, in a natural way, a non-degenerate 2-form on the moduli space Y of stable sheaves on X. The closedness of this 2-form was later proved by the same author in [Mu2], but only for stable vector bundles. A proof of the closedness of Mukai's 2-form valid in general was finally given in [HL]. The construction of the symplectic form on the moduli space Y given by Huybrechts and Lehn differs from the original one proposed by Mukai, and uses the Atiyah class of a universal family of sheaves: this was the inspiration for the constructions presented in the present paper.

Mukai's result was later generalized by S. Kobayashi [K] to the case of the moduli space of simple vector bundles over a compact Kähler manifold X endowed with a holomorphic symplectic structure.

A different proof of the closedness of these so-called "trace forms" can be found in [R]. In this case too, the proof is given only for locally free sheaves.

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Tyurin [Ty] gave another example of the relations between geometric properties of the moduli space Y and those of the space X by proving that the choice of a Poisson structure on an algebraic Poisson surface X determines a Poisson structure on the moduli space Y of stable vector bundles on X. Actually Tyurin did not prove that the Poisson bracket constructed on Y satisfies the Jacobi identity; this is nevertheless true, as we proved later in [B1].

In the present paper we show that similar results hold more generally for differential forms of any degree. More precisely, we prove that there is a natural map which associates to any differential k-form on X or, more generally, to any cohomology class $\sigma \in H^i(X, \Omega_X^{i+k})$, for $i \geq 0$, a closed differential form of degree k on Y. A particularly interesting special case of this result occurs when X is a Calabi-Yau n-fold. In this situation there is a canonical choice (up to scalar multiples) of a n-form on X, hence all moduli spaces of stable sheaves on X are naturally endowed with a closed n-form. This result may be considered as a higher dimensional generalization of Mukai's result.

After a first version of this paper was completed, we came across the paper [KM], by Kuznetsov and Markushevich, where the authors construct closed 2-forms on moduli spaces of sheaves via the Atiyah class of a universal family, using a construction which is essentially a specialization of our construction of closed n-forms to the case $n = 2$. In the introduction of their paper they also claim that their construction can be extended to the case of p-forms. Anyway, the results concerning the existence of closed holomorphic 2-forms contained in [KM] are stated and proved only for the smooth locus of the moduli spaces of stable sheaves on X. This is, in general, a rather strong limitation, because the moduli spaces of stable sheaves on an algebraic variety X of dimension greater than 2 are almost always singular.

In fact, as we shall prove in this paper, the construction of closed holomorphic n-forms on a moduli space of stable sheaves Y can be carried out with no assumption of smoothness of Y.

Let us briefly describe now the organization of the paper. In Section 1 we start by recalling some basic results about cup-products and trace maps for coherent sheaves on a scheme of finite type over a field of characteristic 0. Then we recall the definition of the Atiyah class of a coherent sheaf and we introduce the notion of the local Atiyah class of a flat family of coherent sheaves. In a relative setting, the local Atiyah class turns out to be the natural object to consider, as we shall see. In fact, we shall prove that, when we consider a moduli space Y of stable sheaves on a smooth projective variety X, the local Atiyah class of a universal family of sheaves on Y is well defined even when such a universal family does not exist (which is usually the case). The existence of the local Atiyah class of a universal family of sheaves is the main ingredient we need in order to carry out the construction of closed differential forms on moduli spaces of sheaves.

Finally, we introduce some cohomology classes, called Newton polynomials, constructed by taking the trace of repeated compositions of the Atiyah class with itself. In this case too, in a relative setting, there exists a local version of the Newton polynomials, constructed by using the local Atiyah class.

In Section 2 we use the local Newton polynomials of a universal family of sheaves on the moduli space Y to associate, in a natural way, to any differential k-form on X or, more generally, to any cohomology class $\sigma \in H^i(X, \Omega_X^{i+k})$, for $i \geq 0$, a closed differential form of degree k on Y. This construction actually works even when such a universal family does not exist. If we restrict to the smooth locus of Y, we can also give a more elementary, and more explicit, construction of the closed forms on Y: these can be defined essentially by using the Kodaira-Spencer isomorphism $T_E Y \cong \mathrm{Ext}^1(E,E)$ and by taking the trace of the Yoneda composition of elements of $\mathrm{Ext}^1(E,E)$. This second construction may turn out to be more useful in questions involving the non-degeneracy of these closed forms.

Finally, we end the section with some remarks concerning the possible applications of our results to moduli spaces of sheaves on smooth Calabi-Yau manifolds.

1. Preliminaries

1.1. **Cup-product and trace maps.** In this section we shall briefly recall some standard results about trace maps and cup-products. For more details (and proofs) we refer the reader to [A] or [HL].

Let X be a scheme of finite type over a field k of characteristic 0. In the sequel, whenever we speak of a sheaf on X we shall always mean a sheaf of $\mathcal{O}_X$-modules. Let E be a locally free sheaf of finite rank on X. The usual trace map

$$\mathrm{tr} : \mathcal{E}nd(E) \to \mathcal{O}_X$$

induces natural maps

$$\mathrm{tr} : \mathrm{Ext}^i(E,E) \to H^i(X, \mathcal{O}_X),$$

for any $i \geq 0$. More generally, for any coherent sheaf $\mathcal{F}$ on X, we can define a trace map

$$\mathrm{tr} : \mathcal{H}om(E, E \otimes \mathcal{F}) \to \mathcal{F}.$$

In this case the induced maps on cohomology are

$$\mathrm{tr} : \mathrm{Ext}^i(E, E \otimes \mathcal{F}) \to H^i(X, \mathcal{F}).$$

If E, F and G are three locally free sheaves on X, there is a cup-product (or Yoneda composition) map

$$\mathrm{Ext}^i(F,G) \times \mathrm{Ext}^j(E,F) \xrightarrow{\circ} \mathrm{Ext}^{i+j}(E,G).$$

In the special case $E = F = G$, we find that the composition of the cup-product and trace maps is graded commutative in the following sense: if $\alpha \in \mathrm{Ext}^i(E,E)$

and $\beta \in \operatorname{Ext}^j(E,E)$, then

$$\operatorname{tr}(\alpha \circ \beta) = (-1)^{ij} \operatorname{tr}(\beta \circ \alpha) \tag{1.1}$$

as an element of $H^{i+j}(X, \mathcal{O}_X)$.

Analogous maps can be easily defined when the locally free sheaves E, F and G are replaced by finite complexes $E^{\cdot}$, $F^{\cdot}$ and $G^{\cdot}$ of locally free sheaves of finite rank on X. This fact can be used to define cup-products and trace maps also for coherent sheaves $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ on X, provided that these sheaves admit finite resolutions by locally free sheaves. This is the case, for instance, if X is a smooth projective variety.

Remark 1.1. More generally, cup-products and trace maps can be defined for perfect complexes of sheaves on X (see [SGA6, Exposé I] or [Ill]).

To end this section, we shall recall how the preceding constructions can be carried out in a relative situation.

Let X and Y be two schemes of finite type over k and $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ be Y-flat families of coherent sheaves on X, i.e., coherent sheaves on $X \times Y$, flat over Y. We shall always assume that all these sheaves admit finite locally free resolutions. We shall denote by $p : X \times Y \to X$ and $q : X \times Y \to Y$ the canonical projections and by $\mathcal{E}xt^i_q(\mathcal{E}, \mathcal{F})$ the i-th relative Ext-sheaf, i.e., the i-th derived funtor of $q_* \mathcal{H}om_{\mathcal{O}_{X\times Y}}(\mathcal{E}, \mathcal{F})$.

In this situation there is a relative trace map

$$\operatorname{tr} : \mathcal{E}xt^i_q(\mathcal{E}, \mathcal{E} \otimes \mathcal{F}) \to R^i q_* \mathcal{F}$$

and a relative cup-product

$$\mathcal{E}xt^i_q(\mathcal{F}, \mathcal{G}) \times \mathcal{E}xt^j_q(\mathcal{E}, \mathcal{F}) \xrightarrow{\circ} \mathcal{E}xt^{i+j}_q(\mathcal{E}, \mathcal{G}),$$

satisfying the analogue of (1.1) when $\mathcal{E} = \mathcal{F} = \mathcal{G}$.

1.2. **The Atiyah class.** Let X be a scheme of finite type over k and $\mathcal{E}$ be a coherent sheaf on X which admits a finite locally free resolution. Let us denote by $\mathcal{P}^1_X(\mathcal{E})$ the sheaf of principal parts of order 1 of sections of $\mathcal{E}$ (cf. [EGA4, 4]). There is a natural exact sequence

$$0 \to \mathcal{E} \otimes \Omega^1_X \to \mathcal{P}^1_X(\mathcal{E}) \to \mathcal{E} \to 0, \tag{1.2}$$

hence $\mathcal{P}^1_X(\mathcal{E})$ can be regarded as an extension of $\mathcal{E}$ by $\mathcal{E} \otimes \Omega^1_X$.

Definition 1.2. The Atiyah class of $\mathcal{E}$ is the class

$$a(\mathcal{E}) \in \operatorname{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega^1_X)$$

corresponding to the extension (1.2).

By taking the trace, we get an element $\operatorname{tr}(a(\mathcal{E})) \in H^1(X, \Omega^1_X)$: this is just the opposite of the first Chern class $c_1(\mathcal{E})$.

Remark 1.3. As in the case of the cup-product and trace, the Atiyah class too can be defined more generally for a perfect complex $\mathcal{E}^{\cdot}$ of sheaves on X (see [Ill]).

If Y is another scheme of finite type over k and $\mathcal{E}$ is a Y-flat family of coherent sheaves on X, admitting a finite locally free resolution, we have a global Atiyah class

$$a(\mathcal{E}) \in \operatorname{Ext}^1_{X\times Y}(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times Y}),$$

defined as above. However, in this relative situation we are more interested in a "local" version of the Atiyah class of $\mathcal{E}$, which we shall denote by $\tilde{a}(\mathcal{E})$.

Definition 1.4. The local Atiyah class of $\mathcal{E}$ is the section

$$\tilde{a}(\mathcal{E}) \in H^0(Y, \mathcal{E}xt^1_q(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times Y}))$$

defined as the image of $a(\mathcal{E})$ under the natural map

$$\operatorname{Ext}^1_{X\times Y}(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times Y}) \to H^0(Y, \mathcal{E}xt^1_q(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times Y})) \tag{1.3}$$

coming from the local-to-global spectral sequence $H^i(Y, \mathcal{E}xt^j_q) \Rightarrow \operatorname{Ext}^{i+j}_{X\times Y}$.

Remark 1.5. The sheaf $\mathcal{E}xt^1_q(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times Y})$ is the sheaf on Y associated to the presheaf

$$Y \supseteq U \mapsto \operatorname{Ext}^1_{X\times U}(\mathcal{E}|_{X\times U}, (\mathcal{E} \otimes \Omega^1_{X\times Y})|_{X\times U}).$$

The map (1.3) is the natural map sending a global section of a presheaf to a global section of the associated sheaf.

We shall now explain why, in a relative setting, the "correct" object to consider is not the global Atiyah class $a(\mathcal{E})$ but the local one $\tilde{a}(\mathcal{E})$.

First of all, let us recall that two Y-flat families $\mathcal{E}$ and $\mathcal{F}$ of coherent sheaves on X are said to be equivalent if $\mathcal{F} \cong \mathcal{E} \otimes_{\mathcal{O}_{X\times Y}} q^*L$, for some invertible sheaf L on Y. Since the Atiyah class of a tensor product of sheaves is given by

$$a(\mathcal{E}_1 \otimes \mathcal{E}_2) = a(\mathcal{E}_1) \otimes \mathrm{id}_{\mathcal{E}_2} + \mathrm{id}_{\mathcal{E}_1} \otimes a(\mathcal{E}_2),$$

it follows that $a(\mathcal{F}) = a(\mathcal{E} \otimes_{\mathcal{O}_{X\times Y}} q^*L) \neq a(\mathcal{E})$, in general. However, if we consider the local Atiyah classes, we have:

Lemma 1.6. *Let $\mathcal{E}$ and $\mathcal{F}$ be two Y-flat families of coherent sheaves on X. If $\mathcal{E}$ and $\mathcal{F}$ are equivalent, then the local Atiyah classes $\tilde{a}(\mathcal{E})$ and $\tilde{a}(\mathcal{F})$ are identified under the natural isomorphism*

$$H^0(Y, \mathcal{E}xt^1_q(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times Y})) \cong H^0(Y, \mathcal{E}xt^1_q(\mathcal{F}, \mathcal{F} \otimes \Omega^1_{X\times Y})). \tag{1.4}$$

Proof. If $\mathcal{E}$ and $\mathcal{F}$ are equivalent we can write $\mathcal{F} \cong \mathcal{E} \otimes q^*L$, for some invertible sheaf L on Y. Since L is locally free of rank one, there is a natural isomorphism of sheaves

$$\mathcal{E}xt^1_q(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times Y}) \cong \mathcal{E}xt^1_q(\mathcal{F}, \mathcal{F} \otimes \Omega^1_{X\times Y}), \tag{1.5}$$

hence, from now on, we shall identify these two sheaves.

By taking the global Atiyah class of $\mathcal{F}$, we have

$$a(\mathcal{F}) = a(\mathcal{E}) \otimes \mathrm{id}_{q^*L} + \mathrm{id}_{\mathcal{E}} \otimes a(q^*L).$$

The difference between $a(\mathcal{F})$ and $a(\mathcal{E})$, under the natural identification (1.4) induced by (1.5), is then given by $\mathrm{id}_{\mathcal{E}} \otimes a(q^*L) = \mathrm{id}_{\mathcal{E}} \otimes q^*(a(L))$, where

$$a(L) \in \mathrm{Ext}^1_Y(L, L \otimes \Omega^1_Y) \cong H^1(Y, \Omega^1_Y).$$

The standard exact sequence associated to the local-to-global spectral sequence for the Ext groups is

$$H^1(Y, q_* \mathcal{H}om(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times Y})) \to \mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times Y}) \to H^0(Y, \mathcal{E}xt^1_q(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times Y})).$$

Now it can be checked that $\mathrm{id}_{\mathcal{E}} \otimes a(q^*L)$ belongs to $H^1(Y, q_* \mathcal{H}om(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times Y}))$, hence its image in $H^0(Y, \mathcal{E}xt^1_q(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times Y}))$ is zero. It follows that $\tilde{a}(\mathcal{F}) = \tilde{a}(\mathcal{E})$ as stated. □

1.3. **The Newton polynomials.** Let X be a scheme of finite type over k and $\mathcal{E}$ be a coherent sheaf on X that admits a finite resolution by locally free sheaves. By repeatedly composing the Atiyah class $a(\mathcal{E}) \in \mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega^1_X)$ with itself, we obtain classes in $\mathrm{Ext}^i(\mathcal{E}, \mathcal{E} \otimes (\Omega^1_X)^{\otimes i})$, for any $i \geq 1$. Then, by composing with the map induced by the natural map $(\Omega^1_X)^{\otimes i} \to \Omega^i_X$, we get classes $a(\mathcal{E})^i \in \mathrm{Ext}^i(\mathcal{E}, \mathcal{E} \otimes \Omega^i_X)$.

We now give the following definition:

Definition 1.7. The i-th Newton polynomial of $\mathcal{E}$ is the cohomology class

$$\gamma^i(\mathcal{E}) = \mathrm{tr}(a(\mathcal{E})^i) \in H^i(X, \Omega^i_X).$$

The Newton polynomials of $\mathcal{E}$ coincide, up to a scalar factor, with the components of the Chern character of $\mathcal{E}$ and, as the Chern classes, they are d-closed. More precisely:

Lemma 1.8. *Let $d : H^i(X, \Omega^i_X) \to H^i(X, \Omega^{i+1}_X)$ be the map induced by the exterior differential $d : \Omega^i_X \to \Omega^{i+1}_X$. Then $d\gamma^i(\mathcal{E}) = 0$.*

For the proof of this result we refer the reader to [HL, Sect. 10.1.6].

In a relative situation, where $\mathcal{E}$ is a Y-flat family of coherent sheaves on X, we have the global Newton polynomials $\gamma^i(\mathcal{E}) \in H^i(X \times Y, \Omega^i_{X\times Y})$ and also their local versions, constructed by using the local Atiyah class of $\mathcal{E}$,

$$\tilde{\gamma}^i(\mathcal{E}) \in H^0(Y, R^i q_*(\Omega^i_{X\times Y})).$$

We note that $\tilde{\gamma}^i(\mathcal{E})$ is the image of $\gamma^i(\mathcal{E})$ under the natural map

$$H^i(X \times Y, \Omega^i_{X\times Y}) \to H^0(Y, R^i q_*(\Omega^i_{X\times Y}))$$

sending a global section of a presheaf to a global section of the associated sheaf.

Exactly as for the global Newton polynomials, the local ones too are d-closed, i.e., we have $d\tilde{\gamma}^i(\mathcal{E}) = 0$.

2. Differential Forms on Moduli Spaces

In this section we shall apply the preceding results in order to construct closed differential forms on moduli spaces of sheaves on a smooth projective variety.

From now on we shall denote by X a smooth projective variety defined over k and we take as Y a moduli space of stable sheaves on X.

For the definition and the construction of moduli spaces of stable sheaves on X we refer to [S]: what we need to know, for the moment, is that moduli spaces of stable sheaves exist and they are quasi-projective varieties (usually singular).

In order to apply the constructions described in the previous section we need to know that finite locally free resolutions of coherent sheaves exist: this is indeed the case since X is a smooth projective variety.

Finally, there is another technical problem we have to deal with: the non-existence of universal families of sheaves. In general, on a moduli space Y of stable sheaves on X there does not exist a universal family of sheaves $\mathcal{E}$, not even locally for the Zariski topology. However, universal families on Y exist locally in the étale topology (or in the usual complex topology, if the base field k is the complex field) [S, Theorem 1.21]. These local universal families are not uniquely determined, they are defined only up to tensoring with the pull-back of an invertible sheaf on Y. Usually these ambiguities prevent the local universal families to glue together to a globally defined one (see [Ma, Theorem 6.11] or [HL, Section 4.6] for numerical conditions ensuring the existence of a global universal family on a moduli space of stable sheaves on X).

Let us denote by $\mathcal{E}_\alpha$ a collection of local universal families, one for each open set U_α of a suitable covering of the moduli space Y (in the appropriate topology). Even if the sheaves $\mathcal{E}_\alpha$ do not glue to define a global univeral family $\mathcal{E}$, the relative Ext-sheaves $\mathcal{E}xt^i_q(\mathcal{E}_\alpha, \mathcal{E}_\alpha)$ and $\mathcal{E}xt^i_q(\mathcal{E}_\alpha, \mathcal{E}_\alpha \otimes \Omega^i_{X\times Y}|_{X\times U_\alpha})$ do indeed glue to define global sheaves on Y. For this reason, by abusing notation, we shall denote these sheaves by $\mathcal{E}xt^i_q(\mathcal{E}, \mathcal{E})$ and $\mathcal{E}xt^i_q(\mathcal{E}, \mathcal{E} \otimes \Omega^i_{X\times Y})$, even if the universal family $\mathcal{E}$ does not exist globally on Y.

Since each $\mathcal{E}_\alpha$ is defined only up to tensoring with the pull-back of an invertible sheaf on U_α, the global Atiyah class $a(\mathcal{E}_\alpha)$ is not well defined. However, by recalling the result of Lemma 1.6, we see that the local Atiyah class

$$\tilde{a}(\mathcal{E}_\alpha) \in H^0(U_\alpha, \mathcal{E}xt^1_q(\mathcal{E}_\alpha, \mathcal{E}_\alpha \otimes \Omega^1_{X\times Y}|_{X\times U_\alpha})) = H^0(U_\alpha, \mathcal{E}xt^1_q(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times Y}))$$

is well defined, since it depends only on the equivalence class of $\mathcal{E}_\alpha$. Now it is easy to check that all these local sections $\tilde{a}(\mathcal{E}_\alpha)$ glue together to define a global section of the sheaf $\mathcal{E}xt^1_q(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times Y})$. Again, by abuse of notation, we shall denote this global section by

$$\tilde{a}(\mathcal{E}) \in H^0(Y, \mathcal{E}xt^1_q(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times Y})),$$

even if the universal family $\mathcal{E}$ does not exist.

Analogous considerations hold also for the Newton polynomials: while the global Newton polynomials $\gamma^i(\mathcal{E})$ are not defined, there are well defined local Newton polynomials

$$\tilde{\gamma}^i(\mathcal{E}) \in H^0(Y, R^i q_*(\Omega^i_{X\times Y})).$$

Now, for any open subset $U \subseteq Y$ we have a direct sum decomposition $\Omega^1_{X\times U} = p^*\Omega^1_X \oplus q^*\Omega^1_U$. Since X is a smooth variety, it follows that there is a Künneth decomposition

$$H^n(X \times U, \Omega^n_{X\times U}) = \bigoplus_{i,j=0}^{n} H^i(X, \Omega^j_X) \otimes H^{n-i}(U, \Omega^{n-j}_U),$$

for every $n \geq 0$. The sheaf on Y associated to the presheaf

$$U \mapsto H^n(X \times U, \Omega^n_{X\times U})$$

is the higher direct image $R^n q_*(\Omega^n_{X\times Y})$ and, from the above Künneth decomposition, we obtain an analogous decomposition of sheaves

$$R^n q_*(\Omega^n_{X\times Y}) = \bigoplus_{i,j=0}^{n} H^i(X, \Omega^j_X) \otimes \mathcal{H}^{n-i}(Y, \Omega^{n-j}_Y),$$

where $\mathcal{H}^{n-i}(Y, \Omega^{n-j}_Y)$ is the sheaf associated to the presheaf

$$U \mapsto H^{n-i}(U, \Omega^{n-j}_U)$$

and where, by abuse of notation, we have written $H^i(X, \Omega^j_X)$ to denote the corresponding constant sheaf on Y.

By using this direct sum decomposition, we obtain a decomposition of the local Newton polynomials:

Definition 2.1. For a Newton polynomial $\tilde{\gamma}^n(\mathcal{E}) \in H^0(Y, R^n q_*(\Omega^n_{X\times Y}))$, we shall write $\tilde{\gamma}^n(\mathcal{E}) = \sum_{i,j} \tilde{\gamma}^n_{i,j}(\mathcal{E})$, where $\tilde{\gamma}^n_{i,j}(\mathcal{E})$ is a global section of the sheaf $H^i(X, \Omega^j_X) \otimes \mathcal{H}^{n-i}(Y, \Omega^{n-j}_Y)$.

We note that the restriction of $\tilde{\gamma}^n_{i,j}(\mathcal{E})$ to a sufficiently small open subset $U \subseteq Y$ gives a section

$$\tilde{\gamma}^n_{i,j}(\mathcal{E})|_U \in H^i(X, \Omega^j_X) \otimes H^{n-i}(U, \Omega^{n-j}_Y|_U).$$

We shall now use the sections $\tilde{\gamma}^n_{i,j}(\mathcal{E})$ to associate, to any differential k-form on X, a closed k-form on the moduli space Y.

Theorem 2.2. *Let $n = \dim X$. For any $k = 1, \dots, n$, there is a natural map*

$$f : H^0(X, \Omega^k_X) \to H^0(Y, \Omega^k_Y).$$

Moreover, for any $\sigma \in H^0(X, \Omega^k_X)$, the corresponding k-form $f(\sigma)$ on Y is d-closed.

Proof. The map f is defined by composing the isomorphism

$$H^0(X,\Omega^k_X) \xrightarrow{\sim} H^n(X,\Omega^{n-k}_X)^*$$

given by Serre duality, with the map

$$H^n(X,\Omega^{n-k}_X)^* \longrightarrow H^0(Y,\Omega^k_Y)$$

induced by multiplication by the section $\tilde{\gamma}^n_{n,n-k}(\mathcal{E})$ of the sheaf $H^n(X,\Omega^{n-k}_X) \otimes \mathcal{H}^0(Y,\Omega^k_Y)$. It only remains to show that, for any $\sigma \in H^0(X,\Omega^k_X)$, we have $d(f(\sigma)) = 0$. The proof is essentially identical to the one given in [HL] for 2-forms. It is an easy consequence of the closedness of the Newton polynomials. We shall briefly explain it here.

Let us write $\tilde{\gamma}^n_{n,n-k}(\mathcal{E}) = \sum_l \alpha_l \otimes \beta_l$, for some $\alpha_l \in H^n(X,\Omega^{n-k}_X)$ and some sections β_l of $\mathcal{H}^0(Y,\Omega^k_Y)$. Then, since the Newton polynomials, and also their Künneth components, are d-closed, we can write

$$0 = d\tilde{\gamma}^n_{n,n-k}(\mathcal{E}) = \sum_l \big(d_X\alpha_l \otimes \beta_l + \alpha_l \otimes d_Y\beta_l\big).$$

Now, since X is a smooth projective variety, we have $d_X\alpha_l = 0$, hence it follows that

$$\sum_l \alpha_l \otimes d_Y\beta_l = 0. \tag{2.1}$$

By recalling the definition of the map f, we can write $f(\sigma) = \sum_l \langle \sigma, \alpha_l\rangle\, \beta_l$, where

$$\langle \sigma, \alpha_l\rangle = \int_X \sigma \wedge \alpha_l$$

is the Serre duality pairing. It follows that $d(f(\sigma)) = \sum_l \langle \sigma, \alpha_l\rangle\, d\beta_l = 0$, by (2.1). □

The argument used in the proof of the preceding theorem actually gives the following, more general, result: for any $i, j = 1, \dots, n$ and any $k \geq \max\{n-i, n-j\}$, there is a natural map

$$H^i(X,\Omega^j_X) \longrightarrow H^{k+i-n}(Y,\Omega^{k+j-n}_Y)$$

obtained by composing the Serre duality isomorphism

$$H^i(X,\Omega^j_X) \xrightarrow{\sim} H^{n-i}(X,\Omega^{n-j}_X)^*$$

with the map

$$H^{n-i}(X,\Omega^{n-j}_X)^* \longrightarrow H^{k+i-n}(Y,\Omega^{k+j-n}_Y)$$

induced by multiplication by the section $\tilde{\gamma}^k_{n-i,n-j}(\mathcal{E})$ of the sheaf $H^{n-i}(X,\Omega^{n-j}_X) \otimes \mathcal{H}^{k+i-n}(Y,\Omega^{k+j-n}_Y)$.

In particular, for $k = n-i$, we obtain a map

$$H^i(X,\Omega^j_X) \longrightarrow H^0(Y,\Omega^{j-i}_Y), \tag{2.2}$$

for any $i \le j$.

It follows that we can construct holomorphic p-forms on Y by starting with elements in $H^i(X, \Omega_X^{i+p})$, for any $i \ge 0$. The closedness of such forms follows, exactly as before, from the closedness of the Newton polynomials.

We remark that a similar construction is used in [KM] to define holomorphic symplectic structures on certain moduli spaces of sheaves on X, in some cases when X does not possess any non-zero holomorphic 2-form.

We can also give a more direct and elementary construction of the p-form $\omega \in H^0(Y, \Omega_Y^p)$ associated to a cohomology class $\sigma \in H^i(X, \Omega_X^{i+p})$ via the map (2.2), at least on the smooth locus of the moduli space Y, by using the identification $T_E Y \cong \operatorname{Ext}^1(E,E)$ given by the Kodaira-Spencer map.

At a smooth point $E \in Y$, the map

$$\omega(E) : \underbrace{T_E Y \times \cdots \times T_E Y}_{p} \to k$$

is obtained as the composition of the following maps: first we consider the map

$$\underbrace{\operatorname{Ext}^1(E,E) \times \cdots \times \operatorname{Ext}^1(E,E)}_{p} \to \operatorname{Ext}^p(E,E) \to H^p(X, \mathcal{O}_X) \tag{2.3}$$

given by the Yoneda composition followed by the trace; then we compose with the map

$$H^p(X, \mathcal{O}_X) \xrightarrow{\sigma} H^{i+p}(X, \Omega_X^{i+p})$$

given by multiplication by $\sigma \in H^i(X, \Omega_X^{i+p})$, and finally we compose with the map

$$H^{i+p}(X, \Omega_X^{i+p}) \xrightarrow{\eta_X^{n-i-p}} H^n(X, \Omega_X^n) \cong k, \tag{2.4}$$

where $\eta_X \in H^1(X, \Omega_X^1)$ denotes the cohomology class of the polarization $\mathcal{O}_X(1)$ (i.e., the cohomology class of the Kähler $(1,1)$-form on X, in the complex analytic setting).

This was essentially the approach followed in [B3] to construct closed differential forms on the smooth locus of the moduli space Y of stable sheaves on X. Using this construction, the closedness of the resulting p-form ω can be proved by a direct computation (it is a consequence of a somewhat more general "closedness of trace forms"). We refer to [B3] for more details.

We shall now end this section with some remarks on the existence and non-degeneracy of these closed differential forms on moduli spaces of sheaves.

As a first example let us take $i = 0$, $p = n = \dim X$ and assume that there exists a non-zero section σ of the canonical line bundle $K_X = \Omega_X^n$. In this case the map (2.4) is the identity, hence the n-form ω_σ on the moduli space Y associated to the section σ is given by the composition of (2.3) with the map

$$H^n(X, \mathcal{O}_X) \xrightarrow{\sigma} H^n(X, K_X) \cong k.$$

Even more interesting is the case when the canonical line bundle of X is trivial, i.e., when X is a smooth Calabi-Yau n-fold. In fact, in this case there is a canonical choice (up to scalars) of the n-form σ on X, namely $\sigma = 1 \in H^0(X, K_X) \cong k$, hence there is a n-form ω_σ on the moduli space Y (defined up to proportionality).

The natural question that arises at this point is to know under what conditions the n-form ω_σ on the moduli space Y is non-degenerate. We recall that, for $n = 2$, i.e., when X is an abelian or a K3 surface, this is always the case [Mu1]. For $n \geq 3$, on the other hand, there is no hope that ω_σ be always non-degenerate; in fact there are examples of moduli spaces of stable sheaves (even stable vector bundles) on a smooth Calabi-Yau n-fold that are isomorphic to projective spaces.

We do not know the answer to this question but, in order to investigate the non-degeneracy of the n-form ω_σ, when X is a Calabi-Yau n-fold or in the more general case of a smooth projective variety X with an effective canonical divisor, it may be helpful to use the following algebraic result (whose proof is elementary):

Proposition 2.3. *Let V be a finite dimensional k-vector space and*

$$\omega : \underbrace{V \times \cdots \times V}_{m} \to k$$

be an alternating (or symmetric) multilinear form. Let us define

$$\tilde{\omega} : \underbrace{V \times \cdots \times V}_{m-1} \to V^*$$

by setting

$$\langle v_1, \tilde{\omega}(v_2, \ldots, v_m) \rangle = \omega(v_1, v_2, \ldots, v_m),$$

for any $v_1, \ldots, v_m \in V$. Then we have

$$\mathrm{Ker}(\omega) = \mathrm{Ker}(\tilde{\omega}^t) = (\mathrm{Im}(\tilde{\omega}))^\perp,$$

where

$$\mathrm{Ker}(\omega) = \{v \in V \mid \omega(v, v_2, \ldots, v_m) = 0, \forall v_2, \ldots, v_m \in V\}$$

and $\tilde{\omega}^t$ is the transpose of $\tilde{\omega}$. Hence ω is non-degenerate if and only if $\tilde{\omega}$ is surjective or, equivalently, if and only if $\tilde{\omega}^t$ is injective.

In order to apply this result to our situation we need the following lemma, which is inspired by a similar result proven in [T] (and whose proof can be found in [B3]):

Lemma 2.4. *Let $m = n = \dim X$ and let $\sigma \in H^0(X, K_X)$. Let $E \in Y$ be a smooth point and, using the notations of the preceding proposition, let us set $V = \mathrm{Ext}^1(E, E)$ and $\omega = \omega_\sigma(E)$. Then, by Serre duality, we have $V^* \cong \mathrm{Ext}^{n-1}(E, E \otimes K_X)$, and the map*

$$\tilde{\omega} : \underbrace{\mathrm{Ext}^1(E, E) \times \cdots \times \mathrm{Ext}^1(E, E)}_{n-1} \to \mathrm{Ext}^{n-1}(E, E \otimes K_X) \tag{2.5}$$

is the composition of the map

$$\mathrm{Ext}^1(E,E)\times\cdots\times\mathrm{Ext}^1(E,E)\to\mathrm{Ext}^{n-1}(E,E)$$

given by the Yoneda product, with the map

$$\mathrm{Ext}^{n-1}(E,E)\xrightarrow{\sigma}\mathrm{Ext}^{n-1}(E,E\otimes K_X)$$

defined by the multiplication by $\sigma\in H^0(X,K_X)$.

In general it seems difficult to investigate the surjectivity of the map $\tilde{\omega}$ in (2.5). Obviously, a necessary condition is that the map

$$\mathrm{Ext}^{n-1}(E,E)\xrightarrow{\sigma}\mathrm{Ext}^{n-1}(E,E\otimes K_X) \tag{2.6}$$

be surjective or, equivalently, that its transpose

$$\mathrm{Ext}^1(E,E)\xrightarrow{\sigma}\mathrm{Ext}^1(E,E\otimes K_X),$$

be injective. By applying the functor $\mathrm{Hom}(E,\cdot)$ to the standard exact sequence (we are now assuming E to be locally free)

$$0\longrightarrow E\xrightarrow{\sigma}E\otimes K_X\longrightarrow E\otimes K_X|_D\longrightarrow 0,$$

where $D\in|K_X|$ is the divisor defined by σ, we see that the map above fits into the long exact sequence

$$\begin{aligned}0\to\mathrm{Hom}(E,E)&\xrightarrow{\sigma}\mathrm{Hom}(E,E\otimes K_X)\to\mathrm{Hom}(E,E\otimes K_X|_D)\\ \to\mathrm{Ext}^1(E,E)&\xrightarrow{\sigma}\mathrm{Ext}^1(E,E\otimes K_X).\end{aligned}$$

From the stability of E it follows that $\mathrm{Hom}(E,E)\cong k$, but it is difficult to get informations on $\mathrm{Hom}(E,E\otimes K_X)$ and $\mathrm{Hom}(E,E\otimes K_X|_D)$, in general. Obviously, this problem simply disappears when X is Calabi-Yau, i.e., when $K_X\cong\mathcal{O}_X$ and $\sigma=1$ (in this case the map (2.6) is the identity).

Remark 2.5. If X is a smooth Calabi-Yau n-fold, it may happen that, for a suitable choice of moduli data, the corresponding moduli space Y of stable sheaves on X has an irreducible component Y' which is smooth, projective and of dimension n. Under these hypotheses, if the restriction of the n-form ω_σ to Y' is non-degenerate, then Y' will be a Calabi-Yau n-fold.

An example of this situation can be found in [T, Theorem 4.23]. In this case $n=3$ and the Calabi-Yau 3-fold X is a K3 fibration over $\mathbb{P}^1$. The moduli space Y is a relative moduli space of stable sheaves on X supported on the fibers (with suitable moduli data). The claim is that Y is again a Calabi-Yau 3-fold. To prove this result, Thomas explicitly constructs a holomorphic 3-form on the moduli space Y and shows that it is non-degenerate.

A similar (and more general) problem has been investigated by T. Bridgeland and A. Maciocia, under the additional assumptions that X is a flat Calabi-Yau fibration over a base S, with fibers of dimension ≤ 2, and Y is a relative moduli space of stable sheaves supported on the fibers of $\pi : X \to S$. We refer to [BM] for details.

Remark 2.6. K. Yoshioka has constructed in [Y] moduli spaces of stable twisted sheaves on a smooth complex projective variety X. These are quasi-projective schemes and can be compactified, in the usual way, by adding S-equivalence classes of semistable twisted sheaves. In greater generality, moduli spaces of twisted sheaves have also been constructed by M. Lieblich in [Li], using the language of algebraic stacks. In any case, it turns out that the tangent space to such a moduli space at a point corresponding to a twisted sheaf E is canonically identified with $\mathrm{Ext}^1(E, E)$. From this fact it should follow immediately that our construction of closed differential forms on moduli spaces of stable sheaves can be generalized, in a straightforward way, to moduli spaces of stable twisted sheaves. When X is a K3 surface, this is explicitly proven in [Y]; in this case, the moduli space of stable twisted sheaves on X has a canonical symplectic structure, just as in the untwisted case.

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