

On Countermodels in Basic Logic

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Abstract

In [3] the tautology problem for Hájek's Basic Logic BL is proved to be co-NP-complete by showing that if a formula φ is not a tautology of BL then there exists an integer $m > 0$, polynomially bounded by the length of φ , such that φ fails to be a tautology in the infinite-valued logic $m\mathbf{L}$ corresponding to the ordinal sum of m copies of the Lukasiewicz t -norm. In this paper we state that if φ is not a tautology of BL then it already fails to be a tautology of a finite set of finite-valued logics, defined by taking the ordinal sum of m copies of k -valued Lukasiewicz logics, for effectively determined integers $m, k > 0$ only depending on polynomial-time computable features of φ . This result allows the definition of a calculus for $m\mathbf{L}$ along the lines of [1, 2], while the analysis of features of functions associated with formulas of $m\mathbf{L}$ constitutes a step toward the characterization of finitely generated free BL-algebras as algebras of $[0, 1]$ -valued functions.

1 Introduction

A t -norm is a binary operation from $[0, 1]^2$ into $[0, 1]$ that is associative, commutative, non-decreasing in both arguments, and has 0 as neutral element and 1 as unit. Given a continuous t -norm $*$, it is possible to define its associated *residuum* as the binary operation $x \rightarrow^* y = \max\{z \mid z * x \leq y\}$. We refer to the monograph [7] for further background. Given any continuous t -norm $*$, the *triangular logic* \mathcal{L}_* is the propositional logic on the connectives \odot and \rightarrow where \odot is interpreted as $*$ and \rightarrow as the associated residuum \rightarrow^* ([3]). When the restriction of $*$ and \rightarrow^* is closed with respect to a finite subset S of $[0, 1]$ we can define finite-valued counterparts of the infinite-valued logic \mathcal{L}_* over the finite set of truth-values S . In particular in this paper we shall focus on t -norms such that the $(n + 1)$ -valued logic $(\mathcal{L}_*)_n$ defined over the set $S_n = \{0, 1/n, 2/n, \dots, (n - 1)/n, 1\}$ exists for any positive integer n .

The tautology problem of a given triangular logic \mathcal{L}_* consists in deciding whether a formula is in the set $Taut(\mathcal{L}_*)$ of all formulas taking value 1 in \mathcal{L}_* under any assignment of propositional variables in $[0, 1]$.

The language of propositional Hájek's Basic Logic (BL, for short) [6] contains the binary connectives \odot and \rightarrow , the constant \perp . Axioms of BL are:

$$\begin{array}{ll}
 \text{A1 } (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \vartheta) \rightarrow (\varphi \rightarrow \vartheta)) & \text{A2 } (\varphi \odot \psi) \rightarrow \varphi \\
 \text{A3 } (\varphi \odot \psi) \rightarrow (\psi \odot \varphi) & \text{A4 } (\varphi \odot (\varphi \rightarrow \psi)) \rightarrow (\psi \odot (\psi \rightarrow \varphi)) \\
 \text{A5a } (\varphi \rightarrow (\psi \rightarrow \vartheta)) \rightarrow ((\varphi \odot \psi) \rightarrow \vartheta) & \text{A5b } ((\varphi \odot \psi) \rightarrow \vartheta) \rightarrow (\varphi \rightarrow (\psi \rightarrow \vartheta)) \\
 \text{A6 } ((\varphi \rightarrow \psi) \rightarrow \vartheta) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \vartheta) \rightarrow \vartheta) & \text{A7 } \perp \rightarrow \varphi
 \end{array}$$

and Modus Ponens is the only inference rule.

BL is the logic of all continuous t -norms and their residua [4]. That is, tautologies of Basic logic are exactly the formulas belonging to $Taut(\mathcal{L}_*)$ for all continuous t -norms $*$.

The class of all algebraic models of BL forms a variety \mathcal{BL} of algebras called BL-algebras. \mathcal{BL} includes properly the class of algebras of continuous t -norms.

We observe that a unary connective of negation can be defined as $\neg\varphi := \varphi \rightarrow \perp$. If we consider \neg as primitive and replace axiom A7 by

$$\text{A7}' \quad \neg(\psi \rightarrow \psi) \rightarrow \varphi$$

we obtain an equivalent axiomatization of BL where the primitive set of connectives is $\{\odot, \rightarrow, \neg\}$ instead of $\{\odot, \rightarrow, \perp\}$. We recall that in each triangular logic and in BL, the lattice operations of minimum and maximum can be introduced as the interpretation of derived connectives respectively denoted by \wedge and \vee .

In the case of some t-norms $*$ the axiomatization and the tautology problem of the corresponding triangular logic \mathcal{L}_* has been successfully addressed: axiomatic systems of Lukasiewicz (L), Product (Π) and Gödel (G) logic (corresponding to Lukasiewicz, product and minimum t-norms) are obtained by adding suitable axioms to A1–A7' ([6]), while the complexity of the tautology problem is shown to be co-NP-complete by showing that for any formula which is not a tautology a suitably small countermodel always exists [9, 6, 3].

Finding a countermodel for a formula φ in \mathcal{L}_* amounts to finding an assignment to propositional variables giving φ a value less than 1. Analogously, finding a countermodel for BL means finding a continuous t-norm $*$ and a countermodel for φ in \mathcal{L}_* .

In [3] it is proved that if a formula φ is not a tautology of BL then there exists a suitably bounded integer $m > 0$ such that φ is not a tautology of the logic $m\mathbf{L}$ corresponding to m ordinal copies of Lukasiewicz t-norm. This result, together with the co-NP containment of Lukasiewicz logic [9], allows to prove that the tautology problem for BL is in co-NP, and co-NP-completeness follows from an easy reduction argument.

In [9] it is shown that when a Lukasiewicz formula φ is not a tautology then φ fails to be a tautology of a finite-valued Lukasiewicz logic with a number of truth-values that is polynomially bounded by the length of φ . The bound given in [9] has been improved in [1], and similar bounds are established also for some other triangular logics [2]. These results allows to use calculi for finite-valued logics to prove whether a formula is a tautology of their infinite-valued counterpart and and to find small countermodels if the formula is not a tautology.

In this paper we shall introduce finite-valued counterparts $m\mathbf{L}_k$ for the logics $m\mathbf{L}$ and we shall state that when a formula is not a tautology of BL then it already fails to be a tautology of some finite-valued logic $m\mathbf{L}_k$ (with $mk + 1$ truth-values). Moreover, we shall give a polynomial bound $b_{m\mathbf{L}}(\varphi)$ on mk in terms of the length and the number of different variables of φ . In terms of countermodels this means that in case φ is not a tautology we can effectively find a countermodel assigning variables of φ in the set S_d for $d \leq b_{m\mathbf{L}}(\varphi)$.

The paper is organized as follows: In the following section we shall introduce $m\mathbf{L}$ and its finitely valued counterparts $m\mathbf{L}_k$. We shall also introduce some other infinitely valued propositional logics and their finitely valued counterparts. All other necessary definitions are given. In Section 3 we focus on the notion of minimal countermodel upper bound and we state some results on the logics introduced in Section 2. Section 4 contains the main results of the paper and it is divided in two parts: in the first one we address the problem of finding finite countermodels for formulas in $m\mathbf{L}$, in the second one we use some features of functions associated with formulas of $m\mathbf{L}$ to determine the number m of pieces in the ordinal sum of Lukasiewicz t-norm needed to obtain countermodels for BL. The desired conclusion follows from putting together these last two results.

2 Preliminaries

In this paper we shall consider some infinitely valued propositional triangular logics \mathcal{L} in which the set of truth values is the unit interval of the real line $[0, 1]$. We shall also

make use of their finitely valued counterparts \mathcal{L}_n (which always exist for the logics we shall deal with), where the set of truth-values is a finite subset of $[0, 1]$ of the form

$$S_n = \left\{ 0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1 \right\},$$

for $1 \leq n \in \mathbb{Z}$ and the interpretation of connectives is restricted to S_n . We explicitly observe that the truth-value set S_n gives a $(n+1)$ -valued logic.

The set of connectives for all the logic we shall consider will always be a subset of $Conn = \{\odot, \rightarrow, \neg, \sim, \perp\}$, where \odot, \rightarrow are binary connectives, \neg, \sim are unary connectives and \perp is a constant. The set of formulas constructed starting from $Conn$ and the propositional variables $X_1, X_2, \dots, X_n, \dots$, will be denoted by $Form$. Henceforth, different occurrences of the same subformula in φ shall be considered as different subformulas.

Let φ be any formula built on the set of connectives $Conn$. By $\text{var}(\varphi)$ we denote the set of variables occurring in φ . For each $X_i \in \text{var}(\varphi)$, let $\#(X_i, \varphi)$ be the number of occurrences of X_i in φ , inductively defined as follows:

- If $\varphi = X_i$ then $\#(X_i, \varphi) = 1$. If $\varphi = X_j$ for some $j \neq i$, then $\#(X_i, \varphi) = 0$.
- $\#(X_i, \perp) = 0$.
- If \diamond is a unary connective, then $\#(X_i, \diamond\psi) = \#(X_i, \psi)$.
- If $*$ is a binary connective, then $\#(X_i, \psi * \vartheta) = \#(X_i, \psi) + \#(X_i, \vartheta)$.

Then the *length* of φ , that is the total number of occurrences of variables in φ , is given by: $\#(\varphi) := \sum_{X_i \in \text{var}(\varphi)} \#(X_i, \varphi)$.

A propositional logic \mathcal{L} is determined by a set S of truth values, by a subset of $Conn$ and by a set of functions interpreting connectives in $Conn$ as follows: If $*$ is a connective of arity $\nu(*)$, by $*^{\mathcal{L}} : [0, 1]^{\nu(*)} \rightarrow [0, 1]$ we mean the function interpreting it in \mathcal{L} .

Truth tables $\varphi^{\mathcal{L}}$ of formulas φ in \mathcal{L} are inductively given by

$$\begin{aligned} X_i^{\mathcal{L}}(x_1, \dots, x_n) &= x_i \\ \perp^{\mathcal{L}} &= 0 \\ (\diamond\varphi)^{\mathcal{L}}(x_1, \dots, x_n) &= \diamond^{\mathcal{L}}\varphi^{\mathcal{L}}(x_1, \dots, x_n) \\ (\varphi_1 * \varphi_2)^{\mathcal{L}}(x_1, \dots, x_n) &= \varphi_1^{\mathcal{L}}(x_1, \dots, x_n) *^{\mathcal{L}} \varphi_2^{\mathcal{L}}(x_1, \dots, x_n) \end{aligned}$$

where $|\text{var}(\varphi)| = n$ and $(x_1, \dots, x_n) \in S^n$, $\diamond \in \{\neg, \sim\}$ and $*$ $\in \{\oplus, \rightarrow\}$.

Notice that the interpretation of \perp is the function identically 0 in all logics considered in this paper.

A formula φ is a tautology of \mathcal{L} if and only if $\varphi^{\mathcal{L}}$ is the function identically equal to 1.

Lukasiewicz infinite-valued logic \mathbb{L} is obtained adding to BL the axiom of double negation $\neg\neg\varphi \rightarrow \varphi$. Its semantics is given by interpreting \odot by the Lukasiewicz t-norm and \rightarrow by its residuum:

$$x \odot^{\mathbb{L}} y = \max(0, x + y - 1), \quad x \rightarrow^{\mathbb{L}} y = \min(1, 1 - x + y) \quad \text{and} \quad \neg^{\mathbb{L}} x = 1 - x.$$

The infinite-valued logic \mathbb{L}_{\perp} includes the symbol \perp among its primitive connectives. Note that if φ is a formula of \mathbb{L} then it is a tautology of \mathbb{L} if and only if it is a tautology of \mathbb{L}_{\perp} . On the other hand, for any formula ψ of \mathbb{L}_{\perp} , let ψ' be obtained from ψ by substituting each occurrence of \perp by $\neg(\zeta \rightarrow \zeta)$, for ζ an arbitrarily chosen formula. Then ψ is a tautology of \mathbb{L}_{\perp} if and only if ψ' is a tautology of \mathbb{L} .

Gödel infinite-valued logic \mathbb{G} is an extension of BL logic by the additional axiom $\varphi \rightarrow (\varphi \odot \varphi)$ stating idempotency of the t-norm. Formulas of Gödel logic are built from connectives \odot, \rightarrow and \sim (here we use \sim instead of \neg to denote the negation connective,

the reasons for this choice will be made clear when defining $\mathbf{L} + \mathbf{G}$) that are interpreted for every $x, y \in [0, 1]$, by

$$x \odot^{\mathbf{G}} y = \min(x, y), \quad x \rightarrow^{\mathbf{G}} y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x > y, \end{cases} \quad \text{and} \quad \sim^{\mathbf{G}} x = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0. \end{cases}$$

The infinite-valued logic $\mathbf{L} + \mathbf{G}$ is defined from connectives $\odot, \rightarrow, \neg, \sim$ interpreted respectively as $\odot^{\mathbf{L}}, \rightarrow^{\mathbf{L}}, \neg^{\mathbf{L}}, \sim^{\mathbf{G}}$. Gödel implication $\rightarrow^{\mathbf{G}}$ can then be defined as the interpretation of the derived connective $\varphi \Rightarrow \psi := \neg(\neg \sim \neg(\varphi \rightarrow \psi) \odot \neg\psi)$. Analogously we can define Baaz's unary connective Δ as $\Delta\varphi := \sim \neg\varphi$. Recall that $\Delta x = 1$ if $x = 1$ while $\Delta x = 0$ otherwise.

Logic $\mathbf{L}_{\perp} + \mathbf{G}$ is obtained adding \perp as primitive connective to $\mathbf{L} + \mathbf{G}$.

Finite-valued logics: Each logic \mathcal{L} introduced above admits $(n+1)$ -valued counterparts \mathcal{L}_n for each integer $n \geq 1$, that are obtained by restricting interpretation of connectives to the set S_n .

Definition 2.1 Let m be a positive integer number. For every $i = 1, \dots, m-1$ let

$$I_i^m = \left[\frac{i-1}{m}, \frac{i}{m} \right) \quad \text{and} \quad I_m^m = \left[\frac{m-1}{m}, 1 \right]$$

and let $\sigma_i^m : [0, 1] \rightarrow \text{cl}(I_i^m)$ be the *resizing function* defined by

$$\sigma_i^m(x) = \frac{i-1}{m} + \frac{x}{m},$$

where $\text{cl}(S)$ denotes the topological closure of the set S . Note that σ_i^m is a bijective function and for every $y \in \text{cl}(I_i^m)$, $(\sigma_i^m)^{-1}(y) = my - i + 1$. Further, let us denote by τ_{ij}^m the function from $\text{cl}(I_i)$ to $\text{cl}(I_j)$ obtained by the composition $\sigma_j^m(\sigma_i^m)^{-1}$.

For $x, y \in I_i^m$ let

$$x \odot_i^m y = \sigma_i^m((\sigma_i^m)^{-1}(x) \odot^{\mathbf{L}} (\sigma_i^m)^{-1}(y))$$

and

$$x \rightarrow_i^m y = \sigma_i^m((\sigma_i^m)^{-1}(x) \rightarrow^{\mathbf{L}} (\sigma_i^m)^{-1}(y)).$$

The logic $m\mathbf{L}$ (the triangular logic corresponding to the t -norm given by ordinal sum of m copies of Łukasiewicz t -norm) is built from connectives \odot, \rightarrow and \neg interpreted in the following way: for every $x, y \in [0, 1]$

$$x \odot^m y = \begin{cases} x \odot_i^m y & \text{if } x, y \in I_i \\ \min(x, y) & \text{otherwise} \end{cases} \quad x \rightarrow^m y = \begin{cases} 1 & \text{if } x \leq y \\ x \rightarrow_i^m y & \text{if } x, y \in I_i \text{ and } x > y \\ y & \text{otherwise} \end{cases}$$

$$\neg^m x = x \rightarrow^m 0 = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{m} - x & \text{if } 0 < x \leq \frac{1}{m} \\ 0 & \text{otherwise.} \end{cases}$$

Compare this definition with [3, 7, 8].

In this paper we shall deal with the $(mn+1)$ -valued logics $m\mathbf{L}_n$ given by m many copies of \mathbf{L}_n . That is $m\mathbf{L}_n$ is obtained by restricting the interpretation of connectives \odot^m, \rightarrow^m and \neg^m in $m\mathbf{L}_n$ to S_{nm} .

Notice that finite-valued counterparts of $m\mathbf{L}$ are, for any choice of positive integers n_1, \dots, n_m , the logics $\mathbf{L}_{n_1} \uplus \mathbf{L}_{n_2} \uplus \dots \uplus \mathbf{L}_{n_m}$, with $1 + \sum_{i=1}^m n_i$ truth-values, where the i th component in the ordinal sum is a copy (suitably resized) of \mathbf{L}_{n_i} . Existence of all these logics can be easily proved.

Let φ be a formula with $|\text{var}(\varphi)| = n$ and let $\mathcal{L} \in \{\mathbf{L}, \mathbf{L}_{\perp}, \mathbf{G}, \mathbf{L} + \mathbf{G}, \mathbf{L}_{\perp} + \mathbf{G}\} \cup \{m\mathbf{L} \mid 0 < m \in \mathbb{Z}\}$ be an infinite-valued logic. A set Q of polyhedra is *linearly adequate* for φ in \mathcal{L} if it satisfies the following conditions:

- $\bigcup Q = [0, 1]^n$.
- For each $k \in \{1, \dots, n\}$, for each open k -dimensional polyhedron P in Q the function $\varphi^{\mathcal{L}}$ is linear over P .

A *polyhedral complex* C is a set of polyhedra such that

- If $A \in C$ then all faces of A belong to C .
- If $A, B \in C$ then $A \cap B$ is both a face of A and B .

Each polyhedron in C is called a *cell* of C . Let $0 \leq k \leq n$ be integers such that n is the maximum dimension of cells in C . The set of all k -dimensional cells of C is denoted $C^{(k)}$. Then, the set of all vertices of polyhedra in C is denoted $C^{(0)}$. A polyhedral complex C with all of its cells $\subset \mathbb{R}^n$ is *rational* when $C^{(0)} \subset \mathbb{Q}^n$.

By a standard argument every finite set of polyhedra Q with all rational vertices can be transformed, by subdividing a finite number of its polyhedra with splitting hyperplanes, into a rational polyhedral complex, without adding any new vertex. Then from now on we shall deal only with (rational) polyhedral complexes linearly adequate to formulas.

Note that if C is a polyhedral complex linearly adequate for φ in \mathcal{L} , and $\varphi^{\mathcal{L}} = \psi^{\mathcal{L}}$, then C is linearly adequate for ψ .

For each polyhedron P we shall denote by $\text{rel int } P$ its relative interior. Note that each point $\mathbf{p} \in P$ either is a vertex of P or lies in $\text{rel int } F$ for a unique face F of P (recall that P is a face of itself).

If $\mathbf{p} \in ([0, 1] \cap \mathbb{Q})^n$, and $\mathbf{p} = (r_1/s_1, \dots, r_n/s_n)$ where r_i and s_i are in irreducible form, (i.e., $\text{gcd}(r_i, s_i) = 1$ and $s_i > 0$) for every $i = 1, \dots, n$, then the denominator of \mathbf{p} is

$$\text{den}(\mathbf{p}) = \text{lcm}(s_1, \dots, s_n),$$

where lcm denotes lowest common multiple.

If $a_1/b_1 \dots a_n/b_n \in [0, 1]$ are distinct rational numbers in irreducible form, then their *Farey mediant* is the rational number $(a_1 + \dots + a_n)/(b_1 + \dots + b_n)$. The Farey mediant of affinely independent points $\mathbf{x}_1, \dots, \mathbf{x}_u \in \mathbb{Q}^m$ with $m \geq u - 1$, is obtained by coordinatewise application of the Farey mediant to components of $\mathbf{x}_1, \dots, \mathbf{x}_u$.

Note that the Farey mediant of $\mathbf{x}_1, \dots, \mathbf{x}_u$ is a proper (i.e., different from each $\mathbf{x}_1, \dots, \mathbf{x}_u$) convex combination of such points and its denominator is $\leq \sum_{i=1}^u \text{den}(\mathbf{x}_i)$.

3 Bounds on the cardinality of countermodels

For each infinite-valued logic \mathcal{L} introduced in the previous section and for each formulas φ that is not a tautology of \mathcal{L} , we are interested in finding small countermodels for φ in \mathcal{L} . Actually we would like to find a countermodel which assigns to propositional variables values in S_d for d as small as possible. This is equivalent, in our setting, to find a $(d + 1)$ -valued logic \mathcal{L}_d in which φ already fails to be a tautology.

Definition 3.1 A function $b_{\mathcal{L}} : \text{Form} \rightarrow \mathbb{N}$ is an upper bound on the minimal cardinality of countermodels, or a *minimal countermodel upper bound* (m.c.u.b., for short) for \mathcal{L} , if for every formula $\varphi \in \text{Form}$, φ is a tautology in \mathcal{L} if and only if for every integer $0 < n \leq b_{\mathcal{L}}(\varphi)$, φ is a tautology of \mathcal{L}_n .

To verify that a formula φ in at most n variables is not a tautology of \mathcal{L} we have to find a point $\mathbf{x} \in [0, 1]^n$ such that $\varphi^{\mathcal{L}}(\mathbf{x}) < 1$. For Łukasiewicz logic we know [9, 1], that the point \mathbf{x} can be chosen as a vertex of a cell of some rational polyhedral complex linearly adequate to φ in \mathcal{L} . For Gödel logic [2] the lack of continuity of the function $\sim^{\mathbb{G}}$ and of $\rightarrow^{\mathbb{G}}$ implies that our search of the desired \mathbf{x} cannot be restricted to vertices of cells of a rational polyhedral complex linearly adequate to φ in \mathbb{G} . In this case,

for piecewise linearity, we may have to choose \mathbf{x} as an arbitrary point in the relative interior of some cell. We can find such a point as the Farey mediant of the vertices of that cell (observe that given a rational point \mathbf{x} we can always find a rational polyhedral complex linearly adequate for φ in \mathbf{G} having \mathbf{x} among its vertices, since we only need to introduce sufficiently many (boundaries of) cells that may split regions where $\varphi^{\mathbf{G}}$ is linear). Hence, in order to give an estimation of the minimal countermodel upper bound for a formula φ in each of those two logics, we have to single out those points that occur as vertices of cells or arise as Farey mediants of affinely independent sets of vertices in *all* rational polyhedral complexes linearly adequate for φ . Then, the maximum value of denominators of those points yields a minimal countermodel upper bound.

We shall extend this approach also to all the other logics considered in the paper. That is, given \mathcal{L} we shall find functions $b_{\mathcal{L}}$ (and rational linearly adequate polyhedral complexes $C_{\mathcal{L}}$) such that $\text{den}(p) \leq b_{\mathcal{L}}(\varphi)$ holds for each $\mathbf{p} \in C_{\mathcal{L}}^0(\varphi)$. This kind of m.c.u.b. will be called a *vertex upper bound* (v.u.b., for short). From now on we shall focus on v.u.b.'s only.

In the following theorem we collect to this purpose some old and new results.

Theorem 3.2 *Let $n = |\text{var}(\varphi)|$.*

(i) $b_{\mathbf{L}}(\varphi) := \left(\frac{\#\varphi}{n}\right)^n$ *is a v.u.b. for φ in \mathbf{L} .*

(ii) $b_{\mathbf{G}}(\varphi) := n + 1$ *is a v.u.b. for φ in \mathbf{G} .*

(iii) $b_{\mathbf{L}+\mathbf{G}}(\varphi) := (n + 1) \left(\frac{\#\varphi}{n}\right)^n$ *is a v.u.b. for φ in $\mathbf{L} + \mathbf{G}$.*

(iv) *Suppose a v.u.b. $b_{\mathbf{L}}(\varphi)$ for \mathbf{L} is a function non-decreasing both in $\#\varphi$ and in n . Then $b_{\mathbf{L}+\mathbf{G}}(\varphi) := (n + 1)b_{\mathbf{L}}(\varphi)$ is a v.u.b. for φ in $\mathbf{L} + \mathbf{G}$.*

(v) $b_{\mathbf{L}_{\perp}}(\varphi) := b_{\mathbf{L}}(\varphi)$ *is a v.u.b. for φ in \mathbf{L}_{\perp} and $b_{\mathbf{L}_{\perp}+\mathbf{G}}(\varphi) := b_{\mathbf{L}+\mathbf{G}}(\varphi)$ is a v.u.b. for φ in $\mathbf{L}_{\perp} + \mathbf{G}$.*

(i) is proved in [1]. (ii) is well known and can be found in [6]. Clearly (i) and (iv) implies (iii). (iv) will descend immediately from the following Lemma 3.3, after an application of the Farey mediant to vertices of cells to get points in the relative interior of them (this last step is needed for continuity of $\varphi^{\mathbf{L}+\mathbf{G}}$ over closed cells is not granted). (v) follows from Remark 3.6.

Lemma 3.3 *Suppose a v.u.b. $b_{\mathbf{L}}(\varphi)$ for \mathbf{L} is a function non decreasing both in $\#\varphi$ and in $n = |\text{var}(\varphi)|$.*

For any formula φ there is a rational polyhedral complex $C_{\mathbf{L}+\mathbf{G}}(\varphi)$ linearly adequate for φ in $\mathbf{L} + \mathbf{G}$ such that each vertex $\mathbf{p} \in C_{\mathbf{L}+\mathbf{G}}^{(0)}(\varphi)$ has denominator bounded by $b_{\mathbf{L}}(\varphi)$.

Remark 3.4 The v.u.b. in 3.2.(ii) is known to be sharp, and it admits a simple proof not using the geometric machinery of denominators of vertices of cells exploited here for the other logics under consideration. However, this machinery is shown to work also for \mathbf{G} in [2], where it is applied also to have a v.u.b. b_{Π} for product logic, and to v.u.b.'s for logics defined by further combinations of connectives. The proof of 3.2.(iii) was sketched in [2] for a language admitting a second implication \Rightarrow , interpreted as $\rightarrow^{\mathbf{G}}$, as a primitive connective. A simple adaptation of Lemma 3.3 proves (iii) when \Rightarrow is included among the primitive connectives. As far as the m.c.u.b. for \mathbf{L} is concerned, the first author has recently announced that it can be lowered to $b_{\mathbf{L}}(\varphi) \leq \#\varphi$. The method achieving this result is significantly different from the one here concerned and it does not rely directly on denominators of vertices of cells.

Definition 3.5 Let ψ be an occurrence of a subformula of φ and ϑ an arbitrary formula. Then we denote by $\varphi[\psi/\vartheta]$ the formula obtained from φ by *substituting* the single

occurrence ψ with ϑ . The pair $[\psi/\vartheta]$ will denote the *substitution* of the occurrence ψ with the formula ϑ . Let φ^\perp be the formula obtained from φ by repeated applications of substitutions in the set $\{[\sim \perp/\neg\perp], [\neg\neg\perp/\perp], [\sim \neg\perp/\perp], [\perp \odot \psi/\perp], [\psi \odot \perp/\perp], [\perp \rightarrow \psi/\neg\perp], [\psi \rightarrow \perp/\neg\psi], [\neg\perp \odot \psi/\psi], [\psi \odot \neg\perp/\psi], [\psi \rightarrow \neg\perp/\neg\perp], [\neg\perp \rightarrow \psi/\psi] \mid \psi \text{ occurrence of subformula of } \varphi\}$ until either φ^\perp contains no more occurrences of \perp , or φ^\perp is \perp or $\neg\perp$.

Remark 3.6 Observe that the application to φ of the sequence of substitutions in Definition 3.5 terminates after a finite number of steps, in whichever order and to whichever occurrences the substitutions are applied, always producing the same formula φ^\perp . Further, notice that $\#(\varphi^\perp) \leq \#(\varphi)$ and $(\varphi^\perp)^{\mathbb{L}_\perp + \mathbb{G}} = \varphi^{\mathbb{L}_\perp + \mathbb{G}}$.

Example 3.7 Consider the formula $\varphi := (\neg\perp \rightarrow \psi) \odot \sim (\neg\perp \odot (\vartheta \odot \perp))$, where ψ and ϑ are arbitrary formulas containing no occurrences of \perp . Then

$$\begin{aligned} \varphi[\neg\perp \rightarrow \psi/\psi] &= \psi \odot \sim (\neg\perp \odot (\vartheta \odot \perp)) \\ \psi \odot \sim (\neg\perp \odot (\vartheta \odot \perp))[\vartheta \odot \perp/\perp] &= \psi \odot \sim (\neg\perp \odot \perp) \\ \psi \odot \sim (\neg\perp \odot \perp)[\neg\perp \odot \perp/\perp] &= \psi \odot \sim \perp \\ \psi \odot \sim \perp[\sim \perp/\neg\perp] &= \psi \odot \neg\perp \\ \psi \odot \neg\perp[\psi \odot \neg\perp/\psi] &= \psi = \varphi^\perp \end{aligned}$$

Applying substitutions in a different order: $\varphi^\perp = \varphi[\neg\perp \odot (\vartheta \odot \perp)/\vartheta \odot \perp][\vartheta \odot \perp/\perp][\sim \perp/\neg\perp][\neg\perp \rightarrow \psi] \odot \neg\perp/\neg\perp \rightarrow \psi][\neg\perp \rightarrow \psi/\psi] = \psi$.

4 Main results

We fix a formula φ with n variables and we study the function $\varphi^{m\mathbb{L}} : [0, 1]^n \rightarrow [0, 1]$.

Let π be a function from the set of indexes of variables $\{1, \dots, n\}$ to the set $\{1, \dots, m\}$. We will use the following notation

$$D_\pi = I_{\pi(1)}^m \times \dots \times I_{\pi(n)}^m$$

to denote a region of $[0, 1]^n$. Further we denote by D_i the region $I_i^m \times \dots \times I_i^m$ for every $i = 1, \dots, n$. Note that if $i < j$, then every point $(x_1, \dots, x_n) \in D_\pi$ is such that for every $r \in \pi^{-1}(i)$ and $s \in \pi^{-1}(j)$, $x_r \leq x_s$ (with equality only if $i = j - 1$ and $x_s = i/m = x_r$).

We can extend the functions σ_i^m and τ_{ij}^m of Definition 2.1 to the n -dimensional case. Since m is fixed for all the section, we shall write σ_i and τ_{ij} instead of σ_i^m and τ_{ij}^m . For any $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, m\}$ let $\sigma_\pi : [0, 1]^n \rightarrow D_\pi$ be defined by

$$\sigma_\pi(x_1, \dots, x_n) = (\sigma_{\pi(1)}(x_1), \dots, \sigma_{\pi(n)}(x_n))$$

for any $(x_1, \dots, x_n) \in [0, 1]^n$. If π is the function mapping every $j \in \{1, \dots, n\}$ in i , then σ_π will be denoted by $\sigma_{(i)}$.

For any $\pi, \pi' : \{1, \dots, n\} \rightarrow \{1, \dots, m\}$, let $\tau_{\pi\pi'} : D_\pi \rightarrow D_{\pi'}$ be defined by

$$\tau_{\pi\pi'}(x_1, \dots, x_n) = (\tau_{\pi(1)\pi'(1)}(x_1), \dots, \tau_{\pi(n)\pi'(n)}(x_n)).$$

Take two values $x \in I_i \cup \{0, 1\}$ and $y \in I_j \cup \{0, 1\}$ with $i \neq j$. We observe that we can always find a substitution of either x or y with 0 or 1 in the expression $x \odot^m y$ without changing its value. The same applies to $x \rightarrow^m y$. Then, for each $\mathbf{x} \in [0, 1]^n$, the value of $\varphi^{m\mathbb{L}}(\mathbf{x})$ only depends on variables occurring in one set $\pi^{-1}(h)$. This justifies the following definition which will be used to state some lemmas in the sequel.

Definition 4.1 Let φ be a formula with variables among X_1, \dots, X_n . For every $i \in \{1, \dots, m\}$ let $\theta_1^i, \dots, \theta_{i_i}^i$ be all the maximal occurrences of subformulas of φ containing

only variables whose indexes are not in $\pi^{-1}(i)$. For any $\bar{\beta} = (\beta_1, \dots, \beta_{l_i}) \in \{\perp, \neg\perp\}^{l_i}$, we denote by $\varphi_{i,\bar{\beta}}$ the formula $(\varphi[\theta_1^i/\beta_1] \cdots [\theta_{l_i}^i/\beta_{l_i}])^\perp$ obtained from φ by substituting the occurrence θ_j^i with β_j for all $j \in \{1, \dots, l_i\}$, and by taking off the connective \perp as specified in Definition 3.5. Notice that the only variables appearing in $\varphi_{i,\bar{\beta}}$ have indexes in $\pi^{-1}(i)$.

Further, let $k_{i,j}$ be the total number of occurrences of \neg in θ_j^i , and $\bar{\gamma} = (\gamma_1, \dots, \gamma_{k_{i,j}})$ be a $k_{i,j}$ -tuple in $\{\neg, \sim\}^{k_{i,j}}$. Then the formula $(\theta_j^i)^{\bar{\gamma}}$ is obtained from θ_j^i by substituting the h th occurrence of \neg (according to some fixed order, say leftmost first, for concreteness) with γ_h , for each $h \in \{1, \dots, k\}$.

Finally, let φ^\sim denote the formula obtained from φ by substituting every occurrence of \neg with \sim . That is, φ^\sim is the formula obtained from φ by applying the substitution $[\neg\psi/\sim\psi]$ to every occurrence of subformulas of φ of the form $\neg\psi$.

Observe that $\varphi_{i,\bar{\beta}}$ and φ^\sim are always not longer than φ .

Example 4.2 Let $\varphi = (X_3 \odot \neg X_2) \rightarrow \neg((X_2 \odot X_1) \rightarrow ((X_1 \odot X_3) \rightarrow X_1))$ and let $\pi : \{1, 2, 3\} \rightarrow \{1, 2, 3, 4\}$ be defined by $\pi(1) = 1$, $\pi(2) = 1$, $\pi(3) = 4$. Then we have $D_\pi = I_1 \times I_1 \times I_4$ and $\pi^{-1}(1) = \{1, 2\}$, $\pi^{-1}(2) = \pi^{-1}(3) = \emptyset$, $\pi^{-1}(4) = \{3\}$ and

$$\begin{array}{ll} \theta_1^1 = X_3 \text{ as the occurrence in } X_3 \odot \neg X_2 & \theta_2^1 = X_3 \text{ as the occurrence in } X_1 \odot X_3. \\ \theta_1^4 = \neg X_2 & \theta_2^4 = X_2 \odot X_1 \\ \theta_3^4 = X_1 \text{ as the occurrence in } X_1 \odot X_3 & \theta_4^4 = X_1 \text{ as the last occurrence of } X_1. \end{array}$$

Further:

$$\begin{array}{ll} \varphi[\theta_1^4/\neg\perp] &= (X_3 \odot \neg\perp) \rightarrow \neg((X_2 \rightarrow X_1) \rightarrow ((X_1 \odot X_3) \rightarrow X_1)) \\ \varphi[\theta_2^4/\perp] &= (X_3 \odot \neg X_2) \rightarrow \neg(\perp \rightarrow ((X_1 \odot X_3) \rightarrow X_1)) \\ \varphi[\theta_3^4/\perp] &= (X_3 \odot \neg X_2) \rightarrow \neg((X_2 \rightarrow X_1) \rightarrow ((\perp \odot X_3) \rightarrow X_1)) \\ \varphi[\theta_4^4/\perp] &= (X_3 \odot \neg X_2) \rightarrow \neg((X_2 \rightarrow X_1) \rightarrow ((X_1 \odot X_3) \rightarrow \perp)). \end{array}$$

If we put $\bar{\beta} = (\neg\perp, \perp, \perp, \perp)$ then

$$\varphi_{4,\bar{\beta}} = ((X_3 \odot \neg\perp) \rightarrow \neg(\perp \rightarrow ((\perp \odot X_3) \rightarrow \perp)))^\perp = \neg X_3$$

and the only variables appearing in $\varphi_{4,\bar{\beta}}$ have indexes in $\pi^{-1}(1)$ (i.e., X_3). Note also $\varphi_{4,\bar{\beta}}^\sim = \sim X_3$.

4.1 Countermodels for $m\mathbb{L}$

In this section we shall give an estimation of a v.u.b. $b_{m\mathbb{L}}$ for $m\mathbb{L}$. In order to do that, we have to study the function $\varphi^{m\mathbb{L}}$. In the following lemma we establish a relation between formulas of $m\mathbb{L}$ and of $\mathbb{L}_\perp + \mathbb{G}$. For $i = 1, \dots, m$, $j = 1, \dots, l_i$, let $l_i, \bar{\beta}, k_{i,j}, \theta_j^i, (\theta_j^i)^{\bar{\gamma}}$ be as in Definition 4.1.

Lemma 4.3 *Let $S \subseteq D_\pi$ be a k -dimensional cell of a linearly adequate polyhedral complex $C_{m\mathbb{L}}(\varphi)$ for φ in $m\mathbb{L}$, for some $k \in \{0, \dots, n\}$ and some π . Then $\varphi^{m\mathbb{L}} \upharpoonright \text{rel int } S$ depends only on variables (having indexes) in $\pi^{-1}(j)$ for a unique $j \in \{1, \dots, m\}$.*

Lemma 4.4 *If $\mathbf{x} \in D_\pi$ then $\varphi^{m\mathbb{L}}(\mathbf{x}) \in \bigcup_{j=1}^m I_{\pi(j)} \cup \{0, 1\}$. Further, if $\varphi^{m\mathbb{L}}(\mathbf{x}) \in I_i$, there exists $\bar{\beta}$ such that*

$$\varphi^{m\mathbb{L}}(\mathbf{x}) = \sigma_i((\varphi_{i,\bar{\beta}}^\circ)^{\mathbb{L}_\perp + \mathbb{G}}(\sigma_\pi^{-1}(\mathbf{x})))$$

where $\varphi_{1,\bar{\beta}}^\circ = \varphi_{1,\bar{\beta}}$, while $\varphi_{i,\bar{\beta}}^\circ = \varphi_{i,\bar{\beta}}^\sim$, for $i > 1$.

Note the special role played by the set $\pi^{-1}(1)$, due to the different behaviour of negation in the two intervals $[0, 1/m)$ and $[1/m, 1]$.

We are in a position to describe cells of $C_{m\mathbb{L}}(\varphi)$:

Lemma 4.5 Let $C_{\mathbb{L}}(\varphi)$ be a rational polyhedral complex linearly adequate to φ in \mathbb{L} . Then there exists a rational polyhedral complex $C_{m\mathbb{L}}(\varphi)$ linearly adequate to φ in $m\mathbb{L}$ such that:

$$C_{m\mathbb{L}}^{(0)}(\varphi) \cap D_1 = \sigma_{(1)}(C_{\mathbb{L}}^{(0)}(\varphi))$$

Lemma 4.6 For any formula φ of $m\mathbb{L}$, let φ^\sim be defined as in Definition 4.1. Let $C_{\mathbb{L}+\mathbb{G}}(\varphi^\sim)$ be a rational polyhedral complex linearly adequate to φ^\sim in $\mathbb{L}+\mathbb{G}$. Then there exists a rational polyhedral complex $C_{m\mathbb{L}}(\varphi)$ linearly adequate to φ in $m\mathbb{L}$ such that:

$$C_{m\mathbb{L}}^{(0)}(\varphi) \cap D_i = \sigma_{(i)}(C_{\mathbb{L}+\mathbb{G}}^{(0)}(\varphi^\sim)),$$

for each $i \in \{2, \dots, m\}$.

Lemmas 4.5 and 4.6 are special cases of the following Lemma 4.7. Indeed, consider a region D_π not on the main diagonal (that is π is such that there exist at least two indexes h_1, h_2 with $\pi(h_1) \neq \pi(h_2)$). Applying Lemma 4.3 to each k -dimensional cell $S \subseteq D_\pi$ of a suitably determined rational polyhedral complex $C_{m\mathbb{L}}(\varphi)$ linearly adequate to φ , we have that D_π is partitioned by $\{R_1, \dots, R_m\}$, where each R_i is the union of a finite number of cells of $C_{m\mathbb{L}}(\varphi)$ with the property that $\varphi^{m\mathbb{L}} \upharpoonright R_i$ depends only on variables occurring in $\pi^{-1}(i)$. (In Lemmas 4.5 and 4.6 we have that exactly one R_i is not empty and coincides with D_π). Then we can find each vertex of $C_{m\mathbb{L}}^{(0)}(\varphi) \cap D_\pi$ not lying on the border of some R_i , as vertex of a polyhedral complex linearly adequate to some $\varphi_{i,\bar{\beta}}$ in $\mathbb{L}_\perp + \mathbb{G}$, after resizing through σ_π and correctly interpreting negation as \neg or \sim . We need to account for the remaining vertices that may arise at the border of regions R_i : to accomplish this we take the conjunction of $\varphi_{i,\bar{\beta}}^\circ$ with a formula whose interpretation in $\mathbb{L}_\perp + \mathbb{G}$ vanishes outside $\sigma_\pi^{-1}(R_i)$ while is 1 over the subset of $\sigma_\pi^{-1}(R_i)$ where $\sigma_i((\varphi_{i,\bar{\beta}}^\circ)^{\mathbb{L}_\perp + \mathbb{G}}(\sigma_\pi^{-1}(x))) = \varphi^{m\mathbb{L}}(\mathbf{x})$. This formula is obtained as the conjunction of the formulas θ_j^i , which were *pruned off* to build $\varphi_{i,\bar{\beta}}$, suitably combined with Gödel negation \sim and Baaz's Δ .

Lemma 4.7 Let φ be any formula of $m\mathbb{L}$. For each index $i \in \{1, \dots, m\}$, for each $\bar{\beta}, \bar{\gamma}$ as in Definition 4.1 and for each $\diamond \in \{\sim, \Delta, \neg \sim, \neg \Delta\}$, let $C_{i,\bar{\beta},\bar{\gamma},\diamond}$ be a rational polyhedral complex linearly adequate to

$$\psi_{i,\bar{\beta},\bar{\gamma},\diamond} := \varphi_{i,\bar{\beta}}^\circ \odot \bigcirc_{j=1}^{l_i} \diamond (\theta_j^i)^{\bar{\gamma}}$$

in $\mathbb{L}_\perp + \mathbb{G}$, where $\varphi_{1,\bar{\beta}}^\circ = \varphi_{1,\bar{\beta}}$, while $\varphi_{i,\bar{\beta}}^\circ = \varphi_{i,\bar{\beta}}^\sim$, for $i > 1$. Then there exists a rational polyhedral complex $C_{m\mathbb{L}}^{(0)}(\varphi)$ linearly adequate to φ in $m\mathbb{L}$ such that, for each π and for each vertex $\mathbf{p} \in C_{m\mathbb{L}}^{(0)}(\varphi) \cap D_\pi$, there exists an index $i \in \{1, \dots, m\}$ and $\bar{\beta}, \bar{\gamma}, \diamond$ such that

$$\sigma_\pi^{-1}(\mathbf{p}) \in C_{i,\bar{\beta},\bar{\gamma},\diamond}^{(0)}.$$

Theorem 4.8 Suppose a v.u.b. $b_{\mathbb{L}}(\varphi)$ for \mathbb{L} is a function non-decreasing both in $\#\varphi$ and in n . Then $b_{m\mathbb{L}}(\varphi) := m(n+1)b_{\mathbb{L}}(\varphi)$ is a v.u.b. for $m\mathbb{L}$.

Note that each formula $\psi_{i,\bar{\beta},\bar{\gamma},\diamond}$ is not longer than φ . Then Theorem 4.8 follows from Lemma 4.7 and Theorem 3.2.(iv) and (v).

Example 4.9 Consider the formula $\neg(X_1 \odot X_1) \odot X_2$ in $2\mathbb{L}$. Then for each $\mathbf{x} \in D_1$ we have $\varphi^{2\mathbb{L}}(\mathbf{x}) = \sigma_1(\varphi^{\mathbb{L}}(\sigma_1^{-1}(\mathbf{x})))$, while $\varphi^{2\mathbb{L}}(\mathbf{x}) = 0$ for all $\mathbf{x} \in D_2 \cup (I_2 \times I_1)$. The region $D_\pi = I_1 \times I_2$ is split into two cells: for all $(x_1, x_2) \in D_\pi$ with $x_1 \leq 1/4$ we have that $\varphi^{2\mathbb{L}}(x_1, x_2) = x_2$; for all $(x_1, x_2) \in D_\pi$ with $x_1 > 1/4$ we have $\varphi^{2\mathbb{L}}(x_1, x_2) = 1/2 - x_1$, which can be equivalently expressed as $\sigma_1(\neg(X_1 \odot X_1)^{\mathbb{L}_\perp + \mathbb{G}}(\sigma_\pi^{-1}(\mathbf{x}_1)))$. Finally, observe that the vertices $(1/4, 1/2)$ and $(1/4, 1)$ arise as vertices of $C^{\mathbb{L}_\perp + \mathbb{G}}(X_2 \odot \sim \neg(\neg(X_1 \odot X_1)))$, suitably resized through σ_π .

4.2 On the interdependence among the D_π 's

We stipulate an equivalence relation between the regions D_π .

Definition 4.10 Two functions $\pi, \pi' : \{1, \dots, n\} \rightarrow \{1, \dots, m\}$ are equivalent if

- $\pi^{-1}(1) = \pi'^{-1}(1)$,
- For every $i, j \in \{1, \dots, n\}$, $\pi(i) < \pi(j)$ if and only if $\pi'(i) < \pi'(j)$ and $\pi(i) = \pi(j)$ if and only if $\pi'(i) = \pi'(j)$.

Two regions D_π and $D_{\pi'}$ are equivalent if π and π' are equivalent.

Example 4.11 The set $I_4 \times I_2 \times I_3$ is equivalent to $I_6 \times I_3 \times I_5$, but it is not equivalent to any of the following: $I_2 \times I_4 \times I_3$, $I_4 \times I_1 \times I_3$, $I_4 \times I_2 \times I_2$.

Lemma 4.12 *If π and π' are equivalent then for every $\mathbf{x} \in D_\pi$,*

- *if $\varphi^{mL}(\mathbf{x}) \in I_{\pi(i)}$, then $\tau_{\pi(i)\pi'(i)}(\varphi^{mL}(\mathbf{x})) = \varphi^{mL}(\tau_{\pi\pi'}(\mathbf{x}))$,*
- *if $\varphi^{mL}(\mathbf{x}) = b \in \{0, 1\}$, then $\varphi^{mL}(\tau_{\pi\pi'}(\mathbf{x})) = b$.*

Lemma 4.12 states that the behaviour of φ^{mL} over different D_π 's is not independent. Actually, the only thing that matters in this context, besides the different behaviour of negation over I_1 with respect to all other I_i , is the relative ordering of indexes of variables given by each π . When π and π' are equivalent in this sense (see Definition 4.10) then φ^{mL} is *substantially* the same function over D_π and over $D_{\pi'}$, in the sense made precise by Lemma 4.12.

We are ready to sum up our results:

Theorem 4.13 *Let $n = |\text{var}(\varphi)|$. For every $m \geq n + 1$ a formula φ is a tautology of mL if and only if it is a tautology of $(n + 1)L$ (compare with [3, 8]).*

To prove Theorem 4.13 we have to show that any region D_π with some index $\pi(i) > n + 1$ bears no significant new information with respect to the rest of $[0, 1]^n$. More precisely, by Lemma 4.12 this amounts to verifying that for each such π there exists a $\tilde{\pi}$ with all indexes $\tilde{\pi}(j) \leq n + 1$ that is equivalent, in the sense of Definition 4.10, to π .

Observe that the n many variables occurring in φ can be distributed over at most n many non-empty intervals I_i . Since we have to distinguish cases when $\pi^{-1}(1)$ is non-empty from cases when it is empty, we have to consider at most $n + 1$ different indexes, the worst case being $\pi^{-1}(1) = \emptyset$ and all other $\pi^{-1}(j)$ being singletons. We can conclude that each π is equivalent to some $\tilde{\pi}$ such that $\tilde{\pi}^{-1}(j)$ is empty for all $j \in \{n + 2, \dots, m\}$.

Example 4.14 Consider formulas in $3L$ with at most 2 variables. Then the 9 different regions D_π in which $[0, 1]^2$ is subdivided constitutes 6 different equivalence classes in the sense of Definition 4.10:

$$\{D_1\}, \{D_2, D_3\}, \{I_1 \times I_2, I_1 \times I_3\}, \{I_2 \times I_3\}, \{I_2 \times I_1, I_3 \times I_1\}, \{I_3 \times I_2\}.$$

Then in case $\varphi(X_1, X_2)$ is a tautology of $2L$, but not of $3L$, all countermodels must lie in $(I_2 \times I_3) \cup (I_3 \times I_2)$. For instance, the formula $((X_1 \vee X_2) \rightarrow X_2) \rightarrow X_2 \rightarrow (X_1 \vee (\neg X_2 \rightarrow X_2))$ is a $2L$ -tautology but not a $3L$ -tautology, and hence it is not a tautology of BL . All of its countermodels for $3L$ lie in $I_3 \times I_2$. For each $3 \leq k \in \mathbb{Z}$, each one of the k^2 different regions in which $[0, 1]^2$ is subdivided for kL belongs to one of 6 different classes whose representatives are $D_1, D_2, I_1 \times I_2, I_2 \times I_3, I_2 \times I_1, I_3 \times I_2$. If a formula in two variables is a $3L$ -tautology then it is a kL tautology for each $0 < k \in \mathbb{Z}$, and a BL -tautology as well.

From Theorem 4.13 and Theorem 4.8 we have:

Theorem 4.15 *Suppose a v.u.b. $b_{\mathbb{L}}(\varphi)$ for \mathbb{L} is a function non-decreasing both in $\#\varphi$ and in n . Then a formula φ is a tautology of BL if and only if it is a tautology of every logic $(n+1)\mathbb{L}_k$ for all positive integers $k \leq (n+1)b_{\mathbb{L}}(\varphi)$.*

Applying Theorem 3.2.(iii) we can conclude: $k \leq (n+1) \binom{\#\varphi}{n}$.

4.3 Final Remarks

As is well known, if an algebra A in a variety \mathcal{V} generates the variety of all n -generated algebras in \mathcal{V} , then the free algebra of \mathcal{V} on n generators is precisely the subalgebra of A^{A^n} generated by the projections. From Theorem 4.13 it follows that the free BL-algebra BL_n over n generators is given by the set of functions $\varphi^{(n+1)\mathbb{L}}$ equipped with pointwise defined operations. To give a concrete representation of BL_n we have to fully describe the class of functions associated with formulas. Our results in the present paper describe a class of piecewise linear functions properly containing all functions in BL_n . This can be considered as a first step towards the functional representation of BL_n . Future work will be concerned with the identification of all remaining constraints and interdependences existing among the restrictions of $\varphi^{(n+1)\mathbb{L}}$ to the regions D_π . We recall that in [8] there is a full description of BL_1 as an algebra of functions from $[0, 2]$ to $[0, 2]$, which is, modulo resizing to $[0, 1]$ of range and domain, consistent with our results.

As it is shown in [5] one can routinely define a sufficient set of rules to set up sound and complete set-of-signs tableau calculi [5] or multi-component sequent calculi [1, 2] for all finite-valued triangular logics. In particular one can develop a family $\{C_{m,k} \mid 0 < m, k \in \mathbb{Z}\}$ of calculi for the logics $m\mathbb{L}_k$. Adapting the approach in [1, 2], by using the finite-valued calculi $\{C_{m,k}\}$ and by keeping track of the number of occurring variables and of the length of formulas to be proved, as specified in Theorem 4.15, one can straightforwardly define a sequent calculus sound and complete for BL. Details shall be given elsewhere.

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