Divisible MV-algebras as an algebraic model for fuzzy control

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Abstract

In this paper we shall describe a very natural extension of MV-algebras, the divisible MV-algebras (DMV). Such algebraic structures will be used to give a formal description of rules of fuzzy control.

Key words: Łukasiewicz propositional logics, MV-algebras, fuzzy control.

1 Introduction

Fuzzy control is one of the more successfull fields that has been developed in the framework of fuzzy sets. In recent years also logical and mathematical aspects of fuzzy control have been deeply investigated, see for example [9], [12], [2].

In this paper we propose the representation of fuzzy control by means of formulas of Rational Łukasiewicz calculus. This propositional calculus has been defined in [8] in order to cope with the fact that truth tables of Łukasiewicz logic (also known as McNaughton functions) are continuous piecewise linear functions where each piece can only have integer coefficients. In Rational Łukasiewicz logic truth tables of formulas have rational coefficient, hence are universal approximators of continuous functions.

Just as MV-algebras are the algebraic counterpart of Łukasiewicz logic, in [8] DMV-algebras are defined as the algebras of Rational Łukasiewicz logic.

In the second section we shall give all the necessary background on MV-algebras and Łukasiewicz logic. In Section 3 DMV-algebras and Rational

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Łukasiewicz logic are defined and in Section 4 our proposal for a formalization of fuzzy control is given.

2 Preliminaries

MV-algebras have been introduced by Chang [3] in order to prove a completeness theorem for infinite-valued Łukasiewicz calculus. Formulas of Łukasiewicz propositional logic $Ł_\infty$ are built on from the connectives of negation $\neg$ and disjunction $\oplus$, starting from countably many variables $X_1, X_2, \ldots$, in the usual way ([4]).

The semantics of $Ł_\infty$ is given by the following inductive definition:

Definition 2.1 *Each formula $\varphi(X_1, \ldots, X_n)$ is canonically associated with a function $f_\varphi \colon [0,1]^n \to [0,1]$ (the truth table of φ) by the following inductive stipulation:*

- $f_{X_i}(x_1, \ldots, x_n) = x_i$ *= the ith projection.*
- $f_{\neg\varphi} = 1 - f_\varphi$.
- $f_{(\varphi\oplus\psi)} = \min(1, f_\varphi + f_\psi)$.

The function f_φ is said to be *represented by* (or, f_φ *is associated with*) φ. Two formulas φ and ψ are *logically equivalent* iff $f_\varphi = f_\psi$. A formula φ is *valid* in $Ł_\infty$ (or, φ is a *tautology*) iff f_φ is identically 1 over $[0,1]^n$.

It has been proved ([10]) that a function $f : [0,1]^n \to [0,1]$ is associated with a Łukasiewicz formula if and only if it is a continuous piecewise linear function, such that every linear piece has integer coefficients.

An *MV-algebra* is a structure $A = (A, \oplus, \neg, 0, 1)$ satisfying the following equations:

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z$$

$$x \oplus y = y \oplus x$$

$$x \oplus 0 = x$$

$$x \oplus 1 = 1$$

$$\neg 0 = 1$$

$$\neg 1 = 0$$

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$$

The typical example of MV-algebra is the interval $[0,1]$ equipped with operation $x \oplus y = \min(x + y, 1)$ and $\neg x = 1 - x$.

As proved by Chang, boolean algebras coincide with MV-algebras satisfying the additional equation $x \oplus x = x$ (idempotency). Each MV-algebra contains as a subalgebra the two-element boolean algebra $\{0, 1\}$.

The category of MV-algebras has been proved to be equivalent to the category of abelian lattice-ordered groups with strong unit ([4]). In particular it is possible to associate to each MV-algebra A an abelian lattice ordered group $G_A = Z \times A$. Conversely, if G is an abelian lattice-ordered group and if u is a strong unit of G, i.e., $u \in G$ is such that for every $x \in G$ there exists $n \in N$ with $n.x \geq u$, than $A_G = \Gamma(G, u) = \{x \in G \mid 0 \leq x \leq u\}$, equipped with operations $x \oplus y = \min(x + y, u)$ and $\neg x = u - x$, turns out to be an MV-algebra.

3 DMV-algebras

Both the completeness theorem in [3] and the representation theorem in [5] are based on results for the theory of l-groups. In particular the authors use the result that every totally ordered group can be embedded in a totally ordered divisible group, and that quantifier elimination holds for totally ordered divisible groups.

Thus, it seems natural to study algebraic structures that are more directly connected with divisible groups. In order to do so we define the *divisible MV-algebra (DMV-algebra)*

$$A = (A, \oplus, \neg, \{\delta_n\}_{n \in N}, 0, 1)$$

as an algebraic structure such that $A^* = (A, \oplus, \neg, 0, 1)$ is an MV-algebra and the following hold for every $n \in N$:

(D1) $\quad n.\delta_n x = x$

(D2) $\quad \delta_n x \odot (n-1).\delta_n x = 0$

If A is a DMV-algebra, then the MV-algebra A^* satisfies the condition of divisibility, i.e., for every $n \in N$ and for every $x \in A$ there exists $y \in A$ such that $n.y = x$. The MV-algebra A^* is the *MV-reduct* of the DMV-algebra A. On the other hand, if B is a divisible MV-algebra then by (B, δ_n) we shall denote the DMV-algebra obtained from B by the introduction of the new connective δ_n for every $n > 0$.

Example For each $k = 1, 2, \ldots,$ the set

$$Ł_{k+1} = \{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\},$$

equipped with the operations

$$
\begin{aligned}
x \oplus y &= \min\{1, x + y\}, \\
x \odot y &= \max\{0, x + y - 1\}, \\
\neg x &= 1 - x,
\end{aligned}
$$

is a linearly ordered MV-algebra (also called *MV-chain*), but cannot be enriched to a DMV-algebra. The set of all rationals between 0 and 1 where each δ_n is interpreted as division by n, is a DMV-algebra that we shall denote by $(\mathbf{\Gamma}(\mathbb{Q}, 1), \delta_n)$. In this case, Axioms $(D1n)$ and $(D2n)$ state that the sum of n copies of x/n coincides with x.

If A is a linear DMV-algebra, then A^* is a linear MV-algebra and A^* is isomorphic with $\Gamma(G_{A^*}, (1, 0))$. Let $u = (1, 0)$. For every $n \in N$ and for every $x \in [0, u]$ there exists $y \in [0, u]$ such that $n.y = x$. Since u is a strong unit, for every $x \in G$ there exists an integer number n_x such that $n_x.u \leq x < (n_x{+}1).u$. Let $x' = x - n_x.u \in [0, u]$. Then let y' such that $n.y' = x'$ and let u' such that $n.u' = u$. Then the element $n_x.u' + y'$ is such that $n.(n_x.u' + y') = x$, and so the group G_{A^*} is divisible. In general we have the following

Theorem 3.1 *Let A be a DMV-algebra. Then there exists a unique divisible l-group G together with a strong unit u for G such that $A = \{x \in G \mid 0 \leq x \leq u\}$.*

Ideals of a DMV-algebra A are exactly all the ideals of the MV-reduct A^*. A DMV-algebra A is *semisimple* if A^* is semisimple, i.e., if the intersection of maximal ideals of A is the set $\{0\}$. The following is the analogous of the result for MV-algebras:

Theorem 3.2 *A DMV-algebra A is semisimple if and only if A is isomorphic to a DMV-algebra of $[0, 1]$-valued functions defined over some set.*

Rational Łukasiewicz logic is obtained from Łukasiewicz logic by adding new unary operators δ_n, for each $n \in \mathbb{N}$. Interpretation of formulas of Rational Łukasiewicz logics is given by conditions of Definition 2.1 plus the condition

$$
f_{\delta_n \varphi} = \frac{f_{varphi}}{n}.
$$

It is possible to prove the following

Theorem 3.3 *A function $f : [0, 1]^n \to [0, 1]$ is associated with a formula of Rational Łukasiewicz calculus if and only if it is a continuous piecewise linear functions in which every piece has rational coefficients.*

The set of functions of Theorem 3.3, defined over $[0, 1]^n$ and equipped with pointwise operations is the free DMV-algebra over n variables.

Theorem 3.3 permit us to construct, given a set of rational points $R =$

$\{r_1, \ldots, r_n\}$ in the interval $[0,1]$, a family of formulas $\varphi_1, \ldots, \varphi_n$ with one variable such that $f_{\varphi_i}(r_i) = 1$ and $f_{\varphi_i}(r_j) = 0$ for every $j \neq i$. Such formulas will be called a *fuzzy presentation* of R (see Figure 1).

4 Description of fuzzy control

By *control* we mean the set of actions voted to make assume to a controlled variable a determined sequence of values. Fuzzy control [6] gives a method to represent and implement linguistic description of control systems by means of *fuzzy rules*. Different problems arise in the context of fuzzy control: rules can be either given by an expert or calculated starting from a set of examples, or can be obtained by mixing different approaches and using techniques of soft computing (neural networks, genetic algorithms).

In [9] a formal approach to fuzzy control is given, in terms of the predicative basic fuzzy logic. In [2] the fuzzy control is expressed by a formula of the logic ŁΠ defined in [7].

In this paper we shall consider the following abstraction: given n points $(x_i, f(x_i))$ for a function $f : [0,1] \to [0,1]$ we describe a set of rules in terms of Rational Łukasiewicz formulas that gives an approximation of f (up to an error ϵ_n). Function f represents for every pair (x, y) how much the value y is appropriate to the input value x in order to achieve the control of the system.

Let $f : [0,1] \to [0,1]$ be a function and let $T \subseteq [0,1]^2$ be a finite set of couples $(x_i, f(x_i))$ of rational points, with $i = 1, \ldots, n$. Then we can construct by means of Rational Łukasiewicz formulas, the two fuzzy presentations $\{\varphi_i\}_{i=1}^n$ and $\{\psi_i\}_{i=1}^n$ of sets $\{x_1, \ldots, x_n\}$ and $\{f(x_1), \ldots, f(x_n)\}$. In Figure 1 fuzzy presentations for sets $Ł_8 = \{0, 1/8, \ldots, 7/8, 1\}$ and $Ł_8^2 = \{0, 1/64, 4/64, \ldots, 49/64, 1\}$ ($f(x) = x^2$) are given.

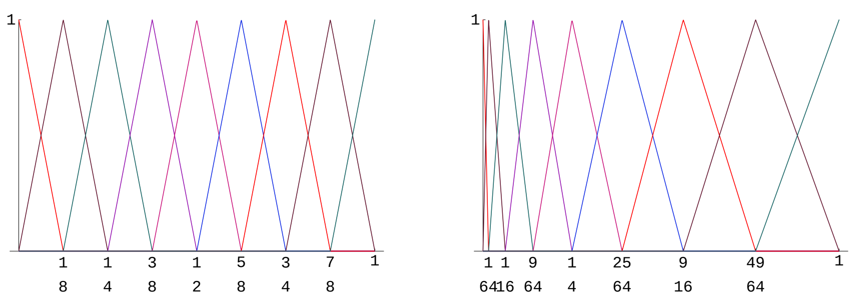


Fig. 1. Examples of fuzzy presentations

We shall consider systems of fuzzy rules of the form:

$$\begin{cases} \textit{Either } X \textit{ is } \varphi_1 \textit{ and } Y \textit{ is } \psi_1 \\ \textit{or } \ldots \\ \textit{or } X \textit{ is } \varphi_n \textit{ and } Y \textit{ is } \psi_n. \end{cases} \tag{1}$$

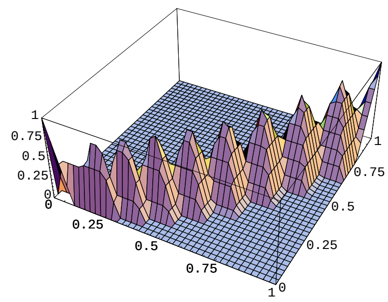
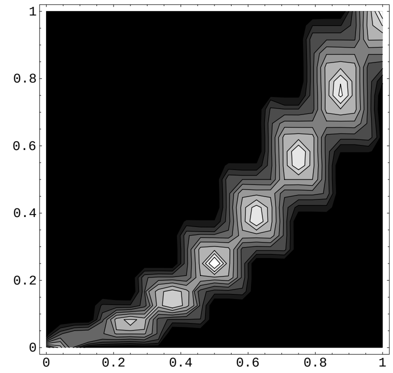
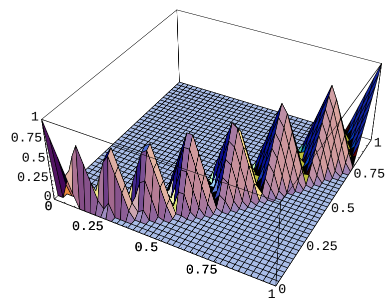
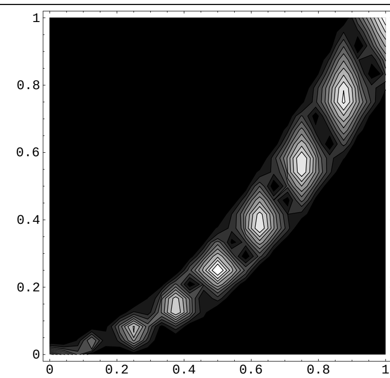
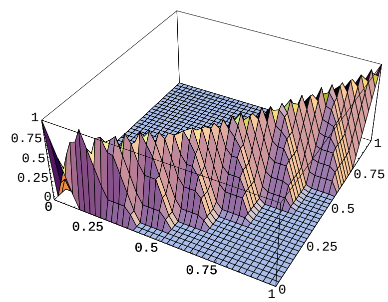
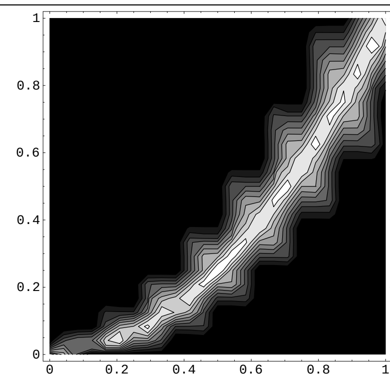
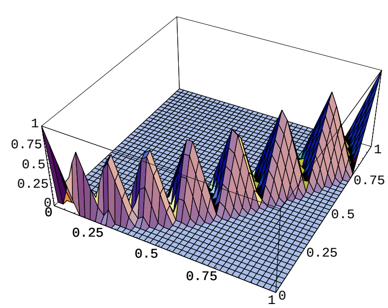
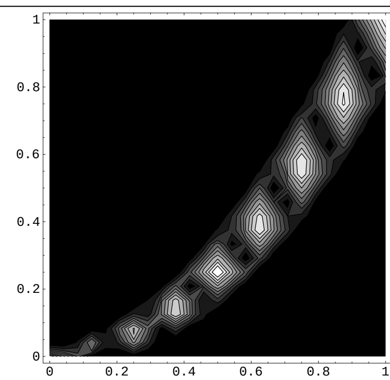
The problem arise on the interpretation of connectives in (1). By using Rational Łukasiewicz logic, four different possibilities arise:

$$\bigvee_{i=1}^{n}(\varphi_i \wedge \psi_i) \qquad \bigvee_{i=1}^{n}(\varphi_i \odot \psi_i)$$
$$\bigoplus_{i=1}^{n}(\varphi_i \wedge \psi_i) \qquad \bigoplus_{i=1}^{n}(\varphi_i \odot \psi_i).$$

Since for every $x, y \in [0,1]$, we have $x \odot y \leq x \wedge y$ and $x \vee y \leq x \oplus y$, then we can partially compare the above formulas. Indeed

$$\bigoplus_{i=1}^{n}(\varphi_i \wedge \psi_i) \geq \bigvee_{i=1}^{n}(\varphi_i \wedge \psi_i) \geq \bigvee_{i=1}^{n}(\varphi_i \odot \psi_i)$$
$$\bigvee_{i=1}^{n}(\varphi_i \odot \psi_i) \leq \bigoplus_{i=1}^{n}(\varphi_i \odot \psi_i) \leq \bigoplus_{i=1}^{n}(\varphi_i \wedge \psi_i).$$

In the following picture, the graphics of the truth tables of such formulas are depicted for the case $f(x) = x^2$ and $n = 8$.

$\bigvee_{i=1}^{n}(\varphi_i \wedge \psi_i)$		
$\bigvee_{i=1}^{n}(\varphi_i \odot \psi_i)$		
$\bigoplus_{i=1}^{n}(\varphi_i \wedge \psi_i)$		
$\bigoplus_{i=1}^{n}(\varphi_i \odot \psi_i)$		

In [2] an algorithm is given to describe fuzzy control by using the logic ŁΠ: in such logic we have beside Łukasiewicz connectives also Product connectives, so that we can express both addition and multiplication (and also subtraction and division). The formula of control is hence given by

$$\phi_f = \bigvee_{i=1}^{n} \varphi_i(X) \cdot \psi_i(Y). \tag{2}$$

Anyway, in (2) a very particular fragment of ŁΠ is used, since the product only occurs between formulas having different variables. We hence propose to use a tensor product between DMV-algebras, defined in the following way.

4.1 Tensor product

Abelian lattice ordered divisible groups are vector lattices over the field of rational numbers, i.e, vector spaces equipped with a structure of lattice. In [11] the tensor product of MV-algebras has been defined. We can extend such a definition to DMV-algebras in the following way:

Definition 4.1 (bimorphism) *If A, B, C are DMV-algebras, a* bimorphism *$\beta : A \times B \to C$ is a function such that $\beta(1, 1) = 1$ and for all $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$ we have*

- $\beta(a, 0) = \beta(0, b) = 0$
- $\beta(a, b_1 \vee b_2) = \beta(a, b_1) \vee \beta(a, b_2)$; $\beta(a_1 \vee a_2, b) = \beta(a_1, b) \vee \beta(a_2, b)$
- $\beta(a, b_1 \wedge b_2) = \beta(a, b_1) \wedge \beta(a, b_2)$; $\beta(a_1 \wedge a_2, b) = \beta(a_1, b) \wedge \beta(a_2, b)$
- *if $b_1 \odot b_2 = 0$ then $\beta(a, b_1) \odot \beta(a, b_2) = 0$ and $\beta(a, b_1 \oplus b_2) = \beta(a, b_1) \oplus \beta(a, b_2)$; symmetrically, if $a_1 \odot a_2 = 0$ then $\beta(a_1, b) \odot \beta(a_2, b) = 0$ and $\beta(a_1 \oplus a_2, b) = \beta(a_1, b) \oplus \beta(a_2, b)$*
- $\beta(\delta_n a, b) = \beta(a, \delta_n b) = \delta_n \beta(a, b)$.

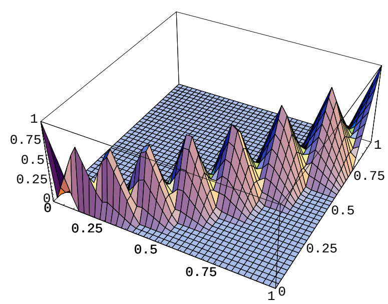
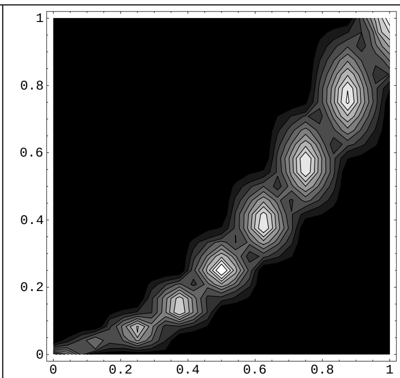
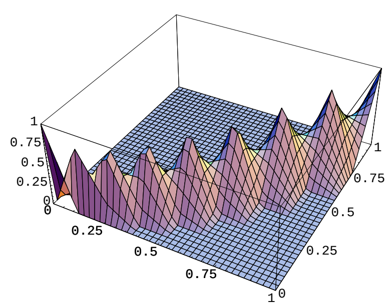
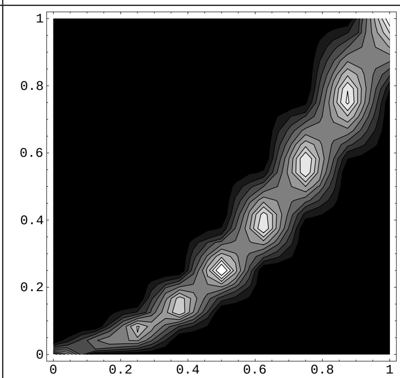
The *tensor product* $A \otimes B$ of DMV-algebras A and B is a DMV-algebra equipped with a bimorphism $\lambda : A \times B \to A \otimes B$ which is universal for all bimorphism. The *semisimple* tensor product for A and B is a DMV-algebra $A \otimes_{ss} B$ with a bimorphism $\lambda_{ss} : A \times B \to A \otimes_{ss} B$ which is universal for the class of semisimple DMV-algebras, equivalently, is the quotient of $A \otimes B$ by the intersection of maximal ideals of $A \times B$.

Then it is possible to prove (see [11]) that if $A \subseteq [0, 1]^X$ and $B \subseteq [0, 1]^Y$ are semisimple, then the semisimple tensor product of A and B is formed by functions $f : X \times Y \to [0, 1]$ such that $f(x, y) = a(x) \cdot b(y)$, where $a \in A$, $b \in B$ and $\cdot$ is the usual product.

When A is the free DMV-algebra over n variables, elements of the semisimple tensor product $A \otimes_{ss} A$ have the form $f_\varphi \cdot f_\psi$.

We shall denoted by Ł⊗Ł the set of functions $f_\varphi \cdot f_\psi$, where φ and ψ are Rational Łukasiewicz formulas. Operations on Ł⊗Ł are defined extending the interpretation of connectives as established by Definition 4.1.

The formula of Equation (2) is hence interpretable by an element of Ł⊗Ł. The function f_{ϕ_f} takes value 1 for every couple $(x, y) \in T$ and represents a *fuzzification* of the function f given the values table T. In the following figure the truth tables of formulas $\bigvee_{i=1}^{n}(\varphi_i \cdot \psi_i)$ and $\bigoplus_{i=1}^{n}(\varphi_i \cdot \psi_i)$ are given, as an example in case $f(x) = x^2$ and $n = 8$.

$\bigvee_{i=1}^{n}(\varphi_i \cdot \psi_i)$		
$\bigoplus_{i=1}^{n}(\varphi_i \cdot \psi_i)$		

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Then an approximation of the function f can be obtained by the defuzzification of f_{ϕ_f} (see for example [1]). Accordingly to the kind of defuzzification used, one can choose to use a The capacity of the control system in Equation (2) to approximate f depends strongly on regularity of the function f, such as continuity and Lipschitz-like hypothesis.

5 Conclusions

We have proposed an algebraic structure, together with a propositional logic, that can furnish a formal framework for fuzzy control. While in this paper we have focused on algebraic and logical properties, future work will be addressed to the comparison of such method with the techniques known in literature.

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